

**Smoothing, Filtering and Prediction:  
Estimating the Past, Present and Future  
Second Edition**  
Garry A. Einicke

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## Preface

Scientists, engineers and the like are a strange lot. Unperturbed by societal norms, they direct their energies to finding better alternatives to existing theories and concocting solutions to unsolved problems. Driven by an insatiable curiosity, they record their observations and crunch the numbers. This tome is about the science of crunching. It's about digging out something of value from the detritus that others tend to leave behind. The described approaches involve constructing models to process the available data. Smoothing entails revisiting historical records in an endeavour to understand something of the past. Filtering refers to estimating what is happening currently, whereas prediction is concerned with hazarding a guess about what might happen next.

The basics of smoothing, filtering and prediction were worked out by Norbert Wiener, Rudolf E. Kalman and Richard S. Bucy *et al* over half a century ago. This book describes the classical techniques together with some more recently developed embellishments for improving performance within applications. Its aims are threefold. First, to present the subject in an accessible way, so that it can serve as a practical guide for undergraduates and newcomers to the field. Second, to differentiate between techniques that satisfy performance criteria versus those relying on heuristics. Third, to draw attention to Wiener's approach for optimal non-causal filtering (or smoothing).

Optimal estimation is routinely taught at a post-graduate level while not necessarily assuming familiarity with prerequisite material or backgrounds in an engineering discipline. That is, the basics of estimation theory can be taught as a standalone subject. In the same way that a vehicle driver does not need to understand the workings of an internal combustion engine or a computer user does not need to be acquainted with its inner workings, implementing an optimal filter is hardly rocket science. Indeed, since the filter recursions are all known – its operation is no different to pushing a button on a calculator. The key to obtaining good estimator performance is developing intimacy with the application at hand, namely, exploiting any available insight, expertise and *a priori* knowledge to model the problem. If the measurement noise is negligible, any number of solutions may suffice. Conversely, if the observations are dominated by measurement noise, the problem may be too hard. Experienced practitioners are able recognise those intermediate sweet-spots where cost-benefits can be realised.

Systems employing optimal techniques pervade our lives. They are embedded within medical diagnosis equipment, communication networks, aircraft avionics, robotics and market forecasting – to name a few. When tasked with new problems, in which information is to be extracted from noisy measurements, one can be faced with a plethora of algorithms and techniques. Understanding the performance of candidate approaches may seem unwieldy and daunting to novices. Therefore, the philosophy here is to present the linear-quadratic-Gaussian results for smoothing, filtering and prediction with accompanying proofs about performance being attained, wherever this is appropriate. Unfortunately, this does

require some maths which trades off accessibility. The treatment is little repetitive and may seem trite, but hopefully it contributes an understanding of the conditions under which solutions can value-add.

Science is an evolving process where what we think we know is continuously updated with refashioned ideas. Although evidence suggests that Babylonian astronomers were able to predict planetary motion, a bewildering variety of Earth and universe models followed. According to lore, ancient Greek philosophers such as Aristotle assumed a geocentric model of the universe and about two centuries later Aristarchus developed a heliocentric version. It is reported that Eratosthenes arrived at a good estimate of the Earth's circumference, yet there was a revival of flat earth beliefs during the middle ages. Not all ideas are welcomed - Galileo was famously incarcerated for knowing too much. Similarly, newly-appearing signal processing techniques compete with old favourites. An aspiration here is to publicise that the oft forgotten approach of Wiener, which in concert with Kalman's, leads to optimal smoothers. The ensuing results contrast with traditional solutions and may not sit well with more orthodox practitioners.

Kalman's optimal filter results were published in the early 1960s and various techniques for smoothing in a state-space framework were developed shortly thereafter. Wiener's optimal smoother solution is less well known, perhaps because it was framed in the frequency domain and described in the archaic language of the day. His work of the 1940s was borne of an analog world where filters were made exclusively of lumped circuit components. At that time, computers referred to people labouring with an abacus or an adding machine – Alan Turing's and John von Neumann's ideas had yet to be realised. In his book, *Extrapolation, Interpolation and Smoothing of Stationary Time Series*, Wiener wrote with little fanfare and dubbed the smoother "unrealisable". The use of the Wiener-Hopf factor allows this smoother to be expressed in a time-domain state-space setting and included alongside other techniques within the designer's toolbox.

A model-based approach is employed throughout where estimation problems are defined in terms of state-space parameters. I recall attending Michael Green's robust control course, where he referred to a distillation column control problem competition, in which a student's robust low-order solution out-performed a senior specialist's optimal high-order solution. It is hoped that this text will equip readers to do similarly, namely: make some simplifying assumptions, apply the standard solutions and back-off from optimality if uncertainties degrade performance.

Both continuous-time and discrete-time techniques are presented. Sometimes the state dynamics and observations may be modelled exactly in continuous-time. In the majority of applications, some discrete-time approximations and processing of sampled data will be required. The material is organised as an eleven-lecture course.

- Chapter 1 introduces some standard continuous-time fare such as the Laplace Transform, stability, adjoints and causality. A completing-the-

square approach is then used to obtain the minimum-mean-square error (or Wiener) filtering solutions.

- Chapter 2 deals with discrete-time minimum-mean-square error filtering. The treatment is somewhat brief since the developments follow analogously from the continuous-time case.
- Chapter 3 describes continuous-time minimum-variance (or Kalman-Bucy) filtering. The filter is found using the conditional mean or least-mean-square-error formula. It is shown for time-invariant problems that the Wiener and Kalman solutions are the same.
- Chapter 4 addresses discrete-time minimum-variance (or Kalman) prediction and filtering. Once again, the optimum conditional mean estimate may be found via the least-mean-square-error approach. Generalisations for missing data, deterministic inputs, correlated noises, direct feedthrough terms, output estimation and equalisation are described.
- Chapter 5 simplifies the discrete-time minimum-variance filtering results for steady-state problems. Discrete-time observability, Riccati equation solution convergence, asymptotic stability and Wiener filter equivalence are discussed.
- Chapter 6 covers the subject of continuous-time smoothing. The main fixed-lag, fixed-point and fixed-interval smoother results are derived. It is shown that the minimum-variance fixed-interval smoother attains the best performance.
- Chapter 7 is about discrete-time smoothing. It is observed that the fixed-point fixed-lag, fixed-interval smoothers outperform the Kalman filter. Once again, the minimum-variance smoother attains the best-possible performance, provided that the underlying assumptions are correct.
- Chapter 8 attends to parameter estimation. As the above-mentioned approaches all rely on knowledge of the underlying model parameters, maximum-likelihood estimation techniques and expectation-maximisation algorithms are described. The addition of a correction term yields unbiased, consistent state-space parameter estimates that attain the Cramer-Rao Lower Bounds.
- Chapter 9 is concerned with robust techniques that accommodate uncertainties within problem specifications. An extra term within the design Riccati equations enables designers to trade-off average error and peak error performance.
- Chapter 10 applies the afore-mentioned linear techniques to nonlinear estimation problems. It is demonstrated that step-wise linearisations can be used within predictors, filters and smoothers, albeit by forsaking optimal performance guarantees.



- Chapter 11 rounds off the course by exploiting knowledge about transition probabilities. HMM, minimum-variance-HMM and high-order minimum-variance-HMM filters and smoothers are derived. The improved performance offered by these techniques needs to be reconciled against the significantly higher calculation overheads.

The foundations are laid in Chapters 1 – 2, which explain minimum-mean-square-error solution construction and asymptotic behaviour. In single-input-single-output cases, finding Wiener filter transfer functions may have appeal. In general, designing Kalman filters is more tractable because solving a Riccati equation is easier than pole-zero cancellation. Kalman filters are needed if the signal models are time-varying. The filtered states can be updated via a one-line recursion but the gain may require to be re-evaluated at each step in time. Extended Kalman filters are contenders if nonlinearities are present. Smoothers are advocated when better performance is desired and some calculation delays can be tolerated. Using additional transition probability information can be advantageous wherever measurements exhibit reoccurring patterns.

This book elaborates on several articles published in *IEEE* journals and I am grateful to my collaborators and the anonymous reviewers who have improved my efforts over the years. Lang White continues to teach and motivate me to this day. The great people at the CSIRO, such as David Hainsworth and George Poropat generously make themselves available to anglicise my engineering jargon. Sometimes posing good questions is helpful, for example, Paul Malcolm once asked “is it stable?” which led down to fruitful paths. During a seminar at HSU, Udo Zoelzer provided the impulse for me to undertake this project. My sources of inspiration include interactions at the CDC meetings - thanks particularly to Dennis Bernstein whose passion for writing has motivated me along the way.

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## Contents

1.	Continuous-Time, Minimum-Mean-Square-Error Filtering .....	1
1.1	Introduction .....	1
1.2	Prerequisites .....	2
1.2.1	Signals .....	2
1.2.2	Elementary Functions Defined on Signals .....	2
1.2.3	Spaces .....	3
1.2.4	Linear Systems .....	3
1.2.5	Polynomial Fraction Systems .....	3
1.2.6	The Laplace Transform of a Signal .....	4
1.2.7	Polynomial Fraction Transfer Functions .....	5
1.2.8	Poles and Zeros .....	6
1.2.9	State-Space Systems .....	6
1.2.10	Euler's Method for Numerical Integration .....	7
1.2.11	State-Space Transfer Function Matrix .....	8
1.2.12	Canonical Realisations .....	9
1.2.13	Asymptotic Stability .....	10
1.2.14	Adjoint Systems .....	10
1.2.15	Causal and Noncausal Systems .....	12
1.2.16	Realising Unstable System Components .....	13
1.2.17	Power Spectral Density .....	14
1.2.18	Spectral Factorisation .....	14
1.3	Minimum-Mean-Square-Error Filtering .....	15
1.3.1	Filter Derivation .....	15
1.3.2	Finding the Causal Part of a Transfer Function .....	17
1.3.3	Minimum-Mean-Square-Error Output Estimation .....	18
1.3.4	Minimum-Mean-Square-Error Input Estimation .....	20
1.4	Chapter Summary .....	21
1.5	Problems .....	23
1.6	Glossary .....	24
1.7	References .....	25
2.	Discrete-Time, Minimum-Mean-Square- Error Filtering .....	27
2.1	Introduction .....	27
2.2	Prerequisites .....	28
2.2.1	Spaces .....	28
2.2.2	Discrete-time Polynomial Fraction Systems .....	29
2.2.3	The Z-Transform of a Discrete-time Sequence .....	29
2.2.4	Polynomial Fraction Transfer Functions .....	29
2.2.5	Poles and Zeros .....	30
2.2.6	Polynomial Fraction Transfer Function Matrix .....	31
2.2.7	State-Space Transfer Function Matrix .....	31
2.2.8	State-Space Realisation .....	31
2.2.9	The Bilinear Approximation .....	32
2.2.10	Discretisation of Continuous-time Systems .....	33

2.2.11	Asymptotic Stability.....	35
2.2.12	Adjoint Systems .....	36
2.2.13	Causal Systems.....	37
2.2.14	Realising Unstable System Components.....	37
2.2.15	Power Spectral Density .....	38
2.2.16	Spectral Factorisation.....	39
2.2.17	Calculating Causal Parts.....	40
2.3	Minimum-Mean-Square-Error Filtering .....	41
2.3.1	Filter Derivation .....	41
2.3.2	Output Estimation .....	43
2.3.3	Input Estimation .....	45
2.4	Chapter Summary .....	46
2.5	Problems .....	48
2.6	Glossary .....	49
2.2	References.....	50
3.	Continuous-Time, Minimum-Variance Filtering .....	53
3.1	Introduction.....	53
3.1	Prerequisites.....	55
3.1.1	The Time-varying Signal Model .....	55
3.1.2	The State Transition Matrix .....	55
3.1.3	The Lyapunov Differential Equation.....	56
3.1.4	Conditional Expectations .....	58
3.2	The Continuous-time Minimum-Variance Filter .....	60
3.2.1	Derivation of the Optimal Filter.....	60
3.2.2	The Riccati Differential Equation .....	62
3.2.3	Including Deterministic Inputs .....	64
3.2.4	Including Correlated Process and Measurement Noise .....	65
3.2.5	Including a Direct Feedthrough Matrix .....	66
3.3	The Continuous-time Steady-State Minimum-Variance Filter .....	67
3.3.1	Riccati Differential Equation Monotonicity .....	67
3.3.2	Observability .....	69
3.3.3	The Algebraic Riccati Equation .....	71
3.3.4	Equivalence of the Wiener and Kalman Filters.....	74
3.4	Chapter Summary .....	76
3.5	Problems .....	77
3.6	Glossary .....	78
3.7	References.....	79
4.	Discrete-Time, Minimum-Variance Filtering .....	83
4.1	Introduction.....	83
4.2	The Time-varying Signal Model.....	84
4.3	The State Prediction Problem.....	85
4.4	The Discrete-time Conditional Mean Estimate .....	86
4.5	Minimum-Variance Prediction.....	87
4.6	Design of the Predictor Gain.....	88
4.7	Minimum-Variance Filtering .....	89
4.8	Design of the Filter Gain.....	90

4.9	The Predictor-Corrector Form.....	91
4.10	The <i>A Posteriori</i> Filter .....	92
4.11	The Information Form.....	92
4.12	Comparison with Recursive Least Squares .....	94
4.13	Repeated Predictions.....	95
4.14	Accommodating Deterministic Inputs .....	97
4.15	Correlated Process and Measurement Noises .....	98
4.16	Including a Direct-Feedthrough Matrix .....	100
4.17	Solution of the General Filtering Problem .....	101
4.18	Hybrid Continuous-Discrete Filtering.....	105
4.19	Chapter Summary .....	106
4.20	Problems .....	107
4.21	Glossary .....	108
4.22	References.....	109
5.	Discrete-Time, Steady-State, Minimum-Variance Filtering .....	111
5.1	Introduction.....	111
5.2	Time-Invariant Filtering Problem .....	112
5.2.1	The Time-Invariant Signal Model.....	112
5.2.2	Application of the Time-Varying Filter Recursions.....	112
5.3	Observability.....	113
5.3.1	The Discrete-time Observability Matrix.....	113
5.3.2	Discrete-time Observability Gramians .....	114
5.4	Riccati Equation Properties.....	116
5.4.1	Monotonicity .....	116
5.4.2	Convergence.....	117
5.5	The Steady-State Minimum-Variance Filter .....	119
5.5.1	State Estimation.....	119
5.5.2	Asymptotic Stability.....	119
5.5.3	Output Estimation .....	120
5.6	Equivalence of the Wiener and Kalman Filters.....	121
5.7	Frequency Weighted Filtering.....	125
5.7.1	Overview .....	125
5.7.2	Problem Definition .....	125
5.7.3	Optimal Filter Solution.....	126
5.7.4	Unweighted Filter Realisation.....	127
5.7.5	Frequency Weighting Method.....	127
5.7.6	Frequency Weighting Design .....	128
5.7.7	Frequency Weighted Output Estimation .....	129
5.7.8	Frequency Weighting Application .....	131
5.8	Chapter Summary .....	132
5.9	Problems .....	134
5.10	Glossary .....	135
5.11	References.....	135
6.	Continuous-time Smoothing .....	137
6.1	Introduction.....	137
6.2	Prerequisites.....	138

6.2.1	Time-varying Adjoint Systems.....	138
6.2.2	Backwards Differential Equations.....	139
6.2.3	Comparison of Riccati Equations.....	140
6.2.4	The Maximum-Likelihood Method.....	141
6.3	Fixed-Point Smoothing.....	143
6.3.1	Problem Definition.....	143
6.3.2	Solution Derivation.....	144
6.3.3	Performance.....	146
6.4	Fixed-Lag Smoothing.....	146
6.4.1	Problem Definition.....	146
6.4.2	Solution Derivation.....	147
6.4.3	Performance.....	148
6.5	Fixed-Interval Smoothing.....	149
6.5.1	Problem Definition.....	149
6.5.2	The Maximum Likelihood Smoother.....	150
6.5.2.1	Solution Derivation.....	150
6.5.2.2	Alternative Form.....	152
6.5.2.3	Performance.....	153
6.5.3	The Fraser-Potter Smoother.....	154
6.5.4	The Minimum-Variance Smoother.....	155
6.5.4.1	Problem Definition.....	155
6.5.4.2	Optimal Unrealisable Solutions.....	156
6.5.4.3	Optimal Realisable Solutions.....	159
6.5.4.4	Performance.....	162
6.5.5	Performance Comparison.....	164
6.6	Chapter Summary.....	165
6.7	Problems.....	167
6.8	Glossary.....	169
6.9	References.....	169
7.	Discrete-Time Minimum-Variance Smoothing.....	173
7.1	Introduction.....	173
7.2	Prerequisites.....	174
7.2.1	Time-varying Adjoint Systems.....	174
7.2.2	Riccati Equation Comparison.....	174
7.3	Fixed-Point Smoothing.....	176
7.3.1	Solution Derivation.....	176
7.3.2	Performance.....	179
7.4	Fixed-Lag Smoothing.....	180
7.4.1	High-order Solution.....	180
7.4.2	Reduced-order Solution.....	181
7.4.3	Performance.....	182
7.5	Fixed-Interval Smoothing.....	182
7.5.1	The Maximum-Likelihood Smoother.....	182
7.5.1.1	Solution Derivation.....	182
7.5.1.2	Alternative Forms.....	184
7.5.1.3	Performance.....	185

7.5.2	The Fraser-Potter Smoother .....	186
7.5.3	Minimum-Variance Smoothers .....	187
7.5.3.1	Optimal Unrealisable Solutions .....	187
7.5.3.2	Non-causal Output Estimation .....	188
7.5.3.3	Causal Output Estimation .....	190
7.5.3.4	Input Estimation .....	190
7.5.3.5	State Estimation .....	191
7.5.3.6	Performance .....	191
7.5.4	Performance Comparison .....	194
7.6	Chapter Summary .....	195
7.7	Problems .....	196
7.8	Glossary .....	197
7.9	References .....	198
8.	Parameter Estimation .....	201
8.1	Introduction .....	201
8.2	Maximum-Likelihood Estimation .....	203
8.2.1	General Method .....	203
8.2.2	State Matrix Estimation .....	205
8.2.3	Variance Estimation .....	207
8.2.4	Cramér-Rao Lower Bound .....	208
8.3	Filtering EM Algorithms .....	212
8.3.1	Background .....	212
8.3.2	Measurement Noise Variance Estimation .....	212
8.3.2.1	EM Algorithm .....	212
8.3.2.2	Properties .....	214
8.3.3	Process Noise Variance Estimation .....	218
8.3.3.1	EM Algorithm .....	218
8.3.3.2	Properties .....	219
8.3.4	State Matrix Estimation .....	223
8.3.4.1	EM Algorithm .....	223
8.3.4.2	Properties .....	225
8.4	Smoothing EM Algorithms .....	227
8.4.1	Process Noise Variance Estimation .....	227
8.4.1.1	EM Algorithm .....	227
8.4.1.2	Properties .....	229
8.4.2	State Matrix Estimation .....	231
8.4.2.2	EM Algorithm .....	231
8.4.2.3	Properties .....	232
8.4.3	Measurement Noise Variance Estimation .....	234
8.5	Unbiased Estimation of State Space Parameters .....	236
8.5.1	Overview .....	236
8.5.2	Problem Definition .....	236
8.5.3	Identification of Model Parameters .....	238
8.5.3.1	Prerequisites .....	238
8.5.3.2	Identification of A with R Known .....	239
8.5.3.3	Identification of Q with R Known .....	241

8.5.3.4	Identification of R with A Known .....	243
8.5.3.5	Identification of unknown A, Q and R.....	244
8.5.4	Example.....	245
8.6	Chapter Summary .....	246
8.7	Problems .....	248
8.8	Glossary .....	249
8.9	References.....	250
9.	Robust Prediction, Filtering and Smoothing.....	253
9.1	Introduction.....	253
9.2	Robust Continuous-time Estimation .....	254
9.2.3	Continuous-Time Bounded Real Lemma .....	254
9.2.4	Continuous-Time $H_\infty$ Filtering .....	257
9.2.4.1	Problem Definition.....	257
9.2.4.2	$H_\infty$ Solution .....	258
9.2.4.3	Properties .....	259
9.2.4.4	Trading-Off $H_\infty$ Performance .....	260
9.2.5	Accommodating Uncertainty .....	262
9.2.5.1	Additive Uncertainty.....	262
9.2.5.2	Multiplicative Uncertainty .....	263
9.2.5.3	Parametric Uncertainty .....	263
9.2.6	Continuous-Time $H_\infty$ Smoothing.....	264
9.2.6.1	Background.....	264
9.2.6.2	Problem Definition.....	265
9.2.6.3	$H_\infty$ Solution .....	265
9.2.7	Performance .....	266
9.2.8	Performance Comparison.....	269
9.3	Robust Discrete-Time Estimation .....	271
9.3.3	Discrete-Time Bounded Real Lemma .....	271
9.3.4	Discrete-Time $H_\infty$ Prediction.....	273
9.3.4.1	Problem Definition.....	273
9.3.4.2	$H_\infty$ Solution .....	273
9.3.4.3	Performance .....	274
9.3.5	Discrete-Time $H_\infty$ Filtering .....	276
9.3.5.1	Problem Definition.....	276
9.3.5.2	$H_\infty$ Solution .....	276
9.3.5.3	Performance .....	277
9.3.6	Solution to the General Filtering Problem.....	279
9.3.7	Discrete-Time $H_\infty$ Smoothing.....	280
9.3.7.1	Problem Definition.....	280
9.3.7.2	$H_\infty$ Solution .....	280
9.3.7.3	$H_\infty$ Performance .....	280
9.3.7.4	Performance Comparison.....	281
9.3.7.5	High SNR and Low SNR Asymptotes .....	282
9.4	Chapter Summary .....	286
9.5	Problems .....	287
9.6	Glossary .....	289

9.7	References.....	289
10.	Nonlinear Prediction, Filtering and Smoothing .....	293
10.1	Introduction.....	293
10.2	Extended Kalman Filtering .....	294
10.2.1	Taylor Series Expansion.....	294
10.2.2	Nonlinear Signal Models.....	295
10.2.3	First-Order Extended Kalman Filter.....	295
10.2.4	Second-Order Extended Kalman Filter .....	297
10.2.5	Third-Order Extended Kalman Filter .....	298
10.3	The Faux Algebraic Riccati Equation Technique .....	300
10.3.1	A Nonlinear Observer .....	300
10.3.2	Gain Selection .....	301
10.3.3	Tracking Multiple Signals .....	301
10.3.4	Stability Conditions.....	303
10.3.5	Applications .....	305
10.4	Robust Extended Kalman Filtering .....	308
10.4.1	Nonlinear Problem Statement .....	308
10.4.2	Robust Solution.....	309
10.5	Nonlinear Smoothing .....	311
10.5.1	Approximate Minimum-Variance Smoother .....	311
10.5.2	Robust Smoother .....	312
10.5.3	Application .....	312
10.6	Constrained Filtering and Smoothing .....	314
10.6.1	Background .....	314
10.6.2	Problem Statement .....	315
10.6.3	Constrained Filtering.....	317
10.6.4	Constrained Smoothing .....	320
10.6	Chapter Summary .....	322
10.7	Problems .....	323
10.8	Glossary .....	325
10.9	References.....	325
11.	Hidden Markov Model Filtering and Smoothing.....	327
11.1	Introduction.....	327
11.2	Prerequisites.....	329
11.2.1	Bayes' Theorem .....	329
11.2.2	Joint Probabilities.....	329
11.2.3	Total Probability Theorem .....	329
11.2.4	Markov Chain.....	330
11.3	HMM Filter and Smoother Derivation.....	331
11.3.1	HMM Filter .....	331
11.3.2	HMM Fixed-Interval Smoother.....	333
11.3.3	Transition Probability Matrix Estimation.....	334
11.3.4	Conditional Observation Likelihood Estimation .....	335
11.4	Minimum-Variance-HMM Filtering and Smoothing.....	336
11.4.1	Motivation .....	336
11.4.2	Prior Literature .....	337



11.4.3	Signal Models.....	338
11.4.3.1	Hidden Markov Model.....	338
11.4.3.2	State-Space Model .....	339
11.4.3.3	Equivalent Markov Model .....	340
11.4.4	Filter and Smoother Design Procedure.....	342
11.4.4.1	Problem Assumptions .....	342
11.4.4.2	Identified Parameters .....	342
11.4.4.3	Model Order Design .....	344
11.4.4.4	Filter and Smoother Recursions .....	345
11.4.5	Application .....	347
11.5	High-Order-Minimum-Variance HMM Filtering and Smoothing .....	350
11.5.1	Overview .....	350
11.5.2	Problem Definition .....	351
11.5.2.1	Kronecker-Product Signal Model .....	351
11.5.2.2	High-Order Markov Chains .....	352
11.5.2.3	Output Estimation .....	354
11.5.3	Optimal Estimation of an $n$ -Step Markov Chain .....	355
11.5.3.1	Filter Realisation .....	355
11.5.3.2	Design Procedure .....	356
11.5.4	Application .....	358
11.6	Chapter Summary .....	360
11.7	Problems .....	362
11.8	Glossary .....	363
11.9	References.....	363

# 1. Continuous-Time, Minimum-Mean-Square-Error Filtering

## 1.1 Introduction

Optimal filtering is concerned with designing the best linear system for recovering data from noisy measurements. It is a model-based approach requiring knowledge of the signal generating system. The signal models, together with the noise statistics are factored into the design in such a way to satisfy an optimality criterion, namely, minimising the square of the error.

A prerequisite technique, the method of least-squares, has its origin in curve fitting. Amid some controversy, Kepler claimed in 1609 that the planets move around the Sun in elliptical orbits [1]. Carl Freidrich Gauss arrived at a better performing method for fitting curves to astronomical observations and predicting planetary trajectories in 1799 [1]. He formally published a least-squares approximation method in 1809 [2], which was developed independently by Adrien-Marie Legendre in 1806 [1]. This technique was famously used by Giusseppe Piazzi to discover and track the asteroid Ceres using a least-squares analysis which was easier than solving Kepler's complicated nonlinear equations of planetary motion [1]. Andrey N. Kolmogorov refined Gauss's theory of least-squares and applied it for the prediction of discrete-time stationary stochastic processes in 1939 [3]. Norbert Wiener, a faculty member at MIT, independently solved analogous continuous-time estimation problems. He worked on defence applications during the Second World War and produced a report entitled *Extrapolation, Interpolation and Smoothing of Stationary Time Series* in 1943. The report was later published as a book in 1949 [4].

Wiener derived two important results, namely, the optimum (non-causal) minimum-mean-square-error solution and the optimum causal minimum-mean-square-error solution [4] – [6]. The optimum causal solution has since become known as the Wiener filter and in the time-invariant case is equivalent to the Kalman filter that was developed subsequently. Wiener pursued practical outcomes and attributed the term “unrealisable filter” to the optimal non-causal solution because “it is not in fact realisable with a finite network of resistances, capacities, and inductances” [4]. Wiener's unrealisable filter is actually the optimum linear smoother.

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“All men by nature desire to know.” *Aristotle*

The optimal Wiener filter is calculated in the frequency domain. Consequently, Section 1.2 touches on some frequency-domain concepts. In particular, the notions of spaces, state-space systems, transfer functions, canonical realisations, stability, causal systems, power spectral density and spectral factorisation are introduced. The Wiener filter is then derived by minimising the square of the error. Three cases are discussed in Section 1.3. First, the solution to general estimation problem is stated. Second, the general estimation results are specialised to output estimation. The optimal input estimation or equalisation solution is then described. An example, demonstrating the recovery of a desired signal from noisy measurements, completes the chapter.

## 1.2 Prerequisites

### 1.2.1 Signals

Let  $(.)^T$  denote the transpose operator. Consider a continuous-time, stochastic (or random) signal  $w(t) = [w_1(t), w_2(t), \dots, w_n(t)]^T$ , with  $w_i(t) \in \mathbb{R}$ ,  $i = 1, \dots, n$ , which belongs to the space  $\mathbb{R}^n$ , or more concisely  $w(t) \in \mathbb{R}^n$ . In general, where  $n > 1$ , the set of  $w(t)$  over all time  $t$  is denoted by the matrix  $w = [w(-\infty), \dots, w(\infty)]$ . Often the focus is on scalar signals, in which case  $w = [w(-\infty), \dots, w(\infty)]$  is a row vector.

### 1.2.2 Elementary Functions Defined on Signals

The inner product  $\langle v, w \rangle$  of two continuous-time signal vectors  $v$  and  $w$  is defined by

$$\langle v, w \rangle = \int_{-\infty}^{\infty} v^T w \, dt. \quad (1)$$

The 2-norm or Euclidean norm of a continuous-time signal vector  $w$ , is defined as  $\|w\|_2 = \sqrt{\langle w, w \rangle} = \sqrt{\int_{-\infty}^{\infty} w^T w \, dt}$ . The square of the 2-norm, that is,  $\|w\|_2^2 = \langle w^T w \rangle = \int_{-\infty}^{\infty} w^T w \, dt$  is commonly known as energy of the signal  $w$ .

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“Scientific discovery consists in the interpretation for our own convenience of a system of existence which has been made with no eye to our convenience at all.” *Norbert Wiener*

### 1.2.3 Spaces

The Lebesgue 2-space, defined as the set of continuous-time signals having finite 2-norm, is denoted by  $\mathcal{L}_2$ . Thus,  $w \in \mathcal{L}_2$  means that the energy of  $w$  is bounded. The following properties hold for 2-norms.

- (i)  $\|v\|_2 = 0 \Rightarrow v = 0$ .
- (ii)  $\|\alpha v\|_2 = |\alpha| \|v\|_2$ .
- (iii)  $\|v + w\|_2 \leq \|v\|_2 + \|w\|_2$ , which is known as the triangle inequality.
- (iv)  $\|vw\|_2 \leq \|v\|_2 \|w\|_2$ .
- (v)  $|\langle v, w \rangle| \leq \|v\|_2 \|w\|_2$ , which is known as the Cauchy-Schwarz inequality.

See [8] for more detailed discussions of spaces and norms.

### 1.2.4 Linear Systems

A linear system is defined as having an output vector which is equal to the value of a linear operator applied to an input vector. That is, the relationships between the output and input vectors are described by linear equations, which may be algebraic, differential or integral. Linear time-domain systems are denoted by upper-case script fonts. Consider two linear systems  $\mathcal{G}: w \rightarrow \mathcal{G}w$ ,  $\mathcal{H}: w \rightarrow \mathcal{H}w$ , that is, they operate on an input  $w$  and produce outputs  $\mathcal{G}w$ ,  $\mathcal{H}w$ . The following properties hold.

$$(\mathcal{G} + \mathcal{H})w = \mathcal{G}w + \mathcal{H}w, \quad (2)$$

$$(\mathcal{GH})w = \mathcal{G}(\mathcal{H}w), \quad (3)$$

$$(\alpha \mathcal{G})w = \alpha (\mathcal{G}w), \quad (4)$$

where  $\alpha \in \mathbb{R}$ . An interpretation of (2) is that a parallel combination of  $\mathcal{G}$  and  $\mathcal{H}$  is equivalent to the system  $\mathcal{G} + \mathcal{H}$ . From (3), a series combination of  $\mathcal{G}$  and  $\mathcal{H}$  is equivalent to the system  $\mathcal{GH}$ . Equation (4) states that scalar amplification of a system is equivalent to scalar amplification of a system's output.

### 1.2.5 Polynomial Fraction Systems

The Wiener filtering results [4] – [6] were originally developed for polynomial fraction descriptions of systems which are described below. Consider an  $n^{\text{th}}$ -order linear, time-invariant system  $\mathcal{G}$  that operates on an input  $w(t) \in \mathbb{R}$  and produces an output  $y(t) \in \mathbb{R}$ . Suppose that the differential equation model for this system is

---

“Science is a way of thinking much more than it is a body of knowledge.” *Carl Edward Sagan*

$$\begin{aligned}
& a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\
& = b_m \frac{d^m w(t)}{dt^m} + b_{m-1} \frac{d^{m-1} w(t)}{dt^{m-1}} + \dots + b_1 \frac{dw(t)}{dt} + b_0 w(t), \quad (5)
\end{aligned}$$

where  $a_0, \dots, a_n$  and  $b_0, \dots, b_m$  are real-valued constant coefficients,  $a_n \neq 0$ , with zero initial conditions. This differential equation can be written in the more compact form

$$\begin{aligned}
& \left( a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \frac{d}{dt} + a_0 \right) y(t) \\
& = \left( b_m \frac{d^m}{dt^m} + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} + \dots + b_1 \frac{d}{dt} + b_0 \right) w(t). \quad (6)
\end{aligned}$$

### 1.2.6 The Laplace Transform of a Signal

The two-sided Laplace transform of a continuous-time signal  $y(t) \in \mathbb{R}$  is denoted by  $Y(s)$  and defined by

$$Y(s) = \int_{-\infty}^{\infty} y(t) e^{-st} dt, \quad (7)$$

where  $s = \sigma + j\omega$  is the Laplace transform variable, in which  $\sigma, \omega \in \mathbb{R}$  and  $j = \sqrt{-1}$ . Given a signal  $y(t)$  with Laplace transform  $Y(s)$ ,  $y(t)$  can be calculated from  $Y(s)$  by taking the inverse Laplace Transform of  $Y(s)$ , which is defined by

$$y(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} Y(s) e^{st} ds. \quad (8)$$

**Theorem 1 Parseval's Theorem [7]:**

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-j\infty}^{j\infty} |Y(s)|^2 ds. \quad (9)$$

**Proof.** Let  $y^H(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} Y^H(s) e^{-st} ds$  and  $Y^H(s)$  denote the Hermitian transpose (or adjoint) of  $y(t)$  and  $Y(s)$ , respectively. The left-hand-side of (9) may be written as

---

“No, no, you're not thinking; you're just being logical.” *Niels Henrik David Bohr*

$$\begin{aligned}
 \int_{-\infty}^{\infty} |y(t)|^2 dt &= \int_{-\infty}^{\infty} y^H(t) y(t) dt \\
 &= \int_{-j\infty}^{j\infty} \frac{1}{2\pi j} \int_{-\infty}^{\infty} Y^H(s) e^{-st} ds y(t) dt \\
 &= \int_{-\infty}^{\infty} \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} y(t) e^{-st} dt Y^H(s) ds \\
 &= \int_{-j\infty}^{j\infty} Y(s) Y^H(s) ds \\
 &= \int_{-j\infty}^{j\infty} |Y(s)|^2 ds .
 \end{aligned}$$

The above theorem is attributed to Parseval whose original work [7] concerned the sums of trigonometric series. An interpretation of (9) is that the energy in the time domain equals the energy in the frequency domain.

## 1.2.7 Polynomial Fraction Transfer Functions

The steady-state response  $y(t) = Y(s)e^{st}$  can be found by applying the complex-exponential input  $w(t) = W(s)e^{st}$  to the terms of (6), which results in

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0) Y(s) e^{st} = (b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0) W(s) e^{st} . \quad (10)$$

Therefore,

$$\begin{aligned}
 Y(s) &= \left[ \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \right] W(s) \\
 &= G(s) W(s) , \quad (11)
 \end{aligned}$$

where

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} . \quad (12)$$

is known as the transfer function of the system. It can be seen from (6) and (12) that the polynomial transfer function coefficients correspond to the system's differential equation coefficients. Thus, knowledge of a system's differential equation is sufficient to identify its transfer function.

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“Nature laughs at the difficulties of integration.” *Pierre-Simon Laplace*

### 1.2.8 Poles and Zeros

The numerator and denominator polynomials of (12) can be factored into  $m$  and  $n$  linear factors, respectively, to give

$$G(s) = \frac{b_m(s - \beta_1)(s - \beta_2) \dots (s - \beta_m)}{a_n(s - \alpha_1)(s - \alpha_2) \dots (s - \alpha_n)}. \quad (13)$$

The numerator of  $G(s)$  is zero when  $s = \beta_i, i = 1 \dots m$ . These values of  $s$  are called the zeros of  $G(s)$ . Zeros in the left-hand-plane are called minimum-phase whereas zeros in the right-hand-plane are called non-minimum phase. The denominator of  $G(s)$  is zero when  $s = \alpha_i, i = 1 \dots n$ . These values of  $s$  are called the poles of  $G(s)$ .

**Example 1.** Consider a system described by the differential equation  $\dot{y}(t) = -y(t) + w(t)$ , in which  $y(t)$  is the output arising from the input  $w(t)$ . From (6) and (12), it follows that the corresponding transfer function is given by  $G(s) = (s + 1)^{-1}$ , which possesses a pole at  $s = -1$ .

The system in Example 1 operates on a single input and produces a single output, which is known as single-input-single-output (SISO) system. Systems operating on multiple inputs and producing multiple outputs are known as multiple-input-multiple-output (MIMO). The corresponding transfer function matrices can be written as equation (14) below, where the components  $G_{ij}(s)$  have the polynomial transfer function form within (12) or (13).

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & & \\ \vdots & & \ddots & \vdots \\ G_{q1}(s) & & \dots & G_{qp}(s) \end{bmatrix}. \quad (14)$$

### 1.2.9 State-Space Systems

Consider a system  $\mathcal{G}$  that operates on an input signal  $w = \{w(t) \in \mathbb{R}^m, t \in [0, T]\}$  and produces an output signal  $y = \{y(t) \in \mathbb{R}^q, t \in [0, T]\}$ , i.e.,  $y = \mathcal{G}w$ . Suppose that the system  $\mathcal{G}$  has a state-space realisation of the form

$$\dot{x}(t) = Ax(t) + Bw(t) \quad (15)$$

$$y(t) = Cx(t) + Dw(t), \quad (16)$$

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“It is important that students bring a certain ragamuffin, barefoot irreverence to their studies, they are not here to worship what is known but to question it.” *Jacob Bronowski*

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{q \times n}$  and  $D \in \mathbb{R}^{q \times m}$ , in which  $x(t) \in \mathbb{R}^n$  is a state vector.  $A = \{a_{i,j}\}$  and  $D = \{d_{i,j}\}$  are respectively known as a state matrix and a direct feed-through matrix. The matrices  $B = \{b_{i,j}\}$  and  $C = \{c_{i,j}\}$  are known as input and output mappings, respectively. For ease of understanding, the state-space model (15) – (16) may be written in the expanded form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} (t) = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} (t) + \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} (t)$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} (t) = \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{q,1} & \cdots & c_{q,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} (t) + \begin{bmatrix} d_{1,1} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots \\ d_{q,1} & \cdots & d_{q,m} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} (t).$$

This system is depicted in Fig. 1.

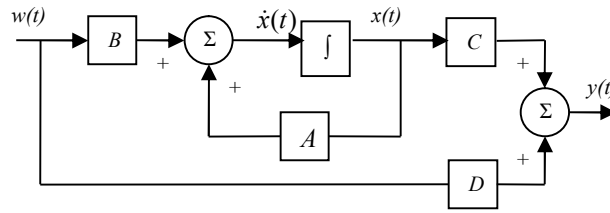


Fig. 1. Continuous-time state-space system.

### 1.2.10 Euler's Method for Numerical Integration

Differential equations of the form (15) could be implemented directly by analog circuits. Digital or software implementations require a method for numerical integration. A first-order numerical integration technique, known as Euler's method, is now described. Suppose that  $x(t)$  is infinitely differentiable and consider its Taylor series expansion in the neighbourhood of  $t_0$

$$\begin{aligned} x(t) &= x(t_0) + \frac{(t-t_0)}{1!} \frac{dx(t_0)}{dt} + \frac{(t-t_0)^2}{2!} \frac{d^2x(t_0)}{dt^2} + \frac{(t-t_0)^3}{3!} \frac{d^3x(t_0)}{dt^3} + \cdots \\ &= x(t_0) + \frac{(t-t_0)}{1!} \dot{x}(t_0) + \frac{(t-t_0)^2}{2!} \ddot{x}(t_0) + \frac{(t-t_0)^3}{3!} \dddot{x}(t_0) + \cdots \end{aligned} \quad (17)$$

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"I have not failed. I've just found 10,000 ways that won't work." *Thomas Alva Edison*



Truncating the series after the first order term yields the approximation  $x(t) = x(t_0) + (t - t_0)\dot{x}(t_0)$ . Defining  $t_k = t_{k-1} + \delta_t$  leads to

$$\begin{aligned} x(t_1) &= x(t_0) + \delta_t \dot{x}(t_0) \\ x(t_2) &= x(t_1) + \delta_t \dot{x}(t_1) \\ &\vdots \\ x(t_{k+1}) &= x(t_k) + \delta_t \dot{x}(t_k). \end{aligned} \quad (18)$$

Thus, the continuous-time linear system (15) could be approximated in discrete-time by iterating

$$\dot{x}(t_{k+1}) = Ax(t_k) + Bw(t_k) \quad (19)$$

and (18) provided that  $\delta_t$  is chosen to be suitably small. Applications of (18) – (19) appear in [9] and in the following example.

**Example 2.** In respect of the continuous-time state evolution (15), consider  $A = -1$ ,  $B = 1$  together with the deterministic input  $w(t) = \sin(t) + \cos(t)$ . The states can be calculated from the known  $w(t)$  using (19) and the difference equation (18). In this case, the state error is given by  $e(t_k) = \sin(t_k) - x(t_k)$ . In particular, root-mean-square-errors of 0.34, 0.031, 0.0025 and 0.00024, were observed for  $\delta_t = 1$ , 0.1, 0.01 and 0.001, respectively. This demonstrates that the first order approximation (18) can be reasonable when  $\delta_t$  is sufficiently small.

### 1.2.11 State-Space Transfer Function Matrix

The transfer function matrix of the state-space system (15) - (16) is defined by

$$G(s) = C(sI - A)^{-1}B + D, \quad (20)$$

in which  $s$  again denotes the Laplace transform variable.

**Example 3.** For a state-space model with  $A = -1$ ,  $B = C = 1$  and  $D = 0$ , the transfer function is  $G(s) = (s + 1)^{-1}$ .

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“Science is everything we understand well enough to explain to a computer. Art is everything else.”  
David Ervin Knuth

**Example 4.** For state-space parameters  $A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $C = [2 \ 5]$

and  $D = 0$ , the use of Cramer's rule, that is,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ ,

yields the transfer function  $G(s) = \frac{(2s+5)}{(s+1)(s+2)} = \frac{1}{(s+1)} + \frac{1}{(s+2)}$ .

**Example 5.** Substituting  $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$  and  $B = C = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  into (20)

results in the transfer function matrix  $G(s) = \begin{bmatrix} \frac{s+2}{s+1} & 0 \\ 0 & \frac{s+3}{s+2} \end{bmatrix}$ .

### 1.2.12 Canonical Realisations

The mapping of a polynomial fraction transfer function (12) to a state-space representation (20) is not unique. Two standard state-space realisations of polynomial fraction transfer functions are described below. Assume that: the transfer function has been expanded into the sum of a direct feed-through term plus a strictly proper transfer function, in which the order of the numerator polynomial is less than the order of the denominator polynomial; and the strictly proper transfer function has been normalised so that  $a_n = 1$ . Under these assumptions, the system can be realised in the controllable canonical form which is parameterised by [10]

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \text{ and } C = [b_m \ b_{m-1} \ \dots \ b_1 \ b_0].$$

The system can be also realised in the observable canonical form which is parameterised by

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“Science might almost be redefined as the process of substituting unimportant questions which can be answered for important questions which cannot.” *Kenneth Ewart Boulding*

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & & 0 \\ \vdots & & & \ddots & 0 \\ -a_1 & & & 0 & 1 \\ -a_0 & 0 & \dots & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} b_m \\ b_{m-1} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} \text{ and } C = [1 \ 0 \ \dots \ 0 \ 0].$$

### 1.2.13 Asymptotic Stability

Consider a continuous-time, linear, time-invariant  $n^{\text{th}}$ -order system  $\mathcal{G}$  that operates on an input  $w$  and produces an output  $y$ . The system  $\mathcal{G}$  is said to be asymptotically stable if the output remains bounded, that is,  $y \in \mathcal{L}_2$ , for any  $w \in \mathcal{L}_2$ . This is also known as bounded-input-bounded-output stability. Two equivalent conditions for  $\mathcal{G}$  to be asymptotically stable are:

- (i) The real part of the eigenvalues of the system's state matrix are in the left-hand-plane, that is, for  $A$  of (20),  $\text{Re}\{\lambda_i(A)\} < 0$ ,  $i = 1 \dots n$ .
- (ii) The real part of the poles of the system's transfer function are in the left-hand-plane, that is, for  $\alpha_i$  of (13),  $\text{Re}\{\alpha_i\} < 0$ ,  $i = 1 \dots n$ .

**Example 6.** A state-space system having  $A = -1$ ,  $B = C = 1$  and  $D = 0$  is stable, since  $\lambda(A) = -1$  is in the left-hand-plane. Equivalently, the corresponding transfer function  $G(s) = (s + 1)^{-1}$  has a pole at  $s = -1$  which is in the left-hand-plane and so the system is stable. Conversely, the transfer function  $G^T(-s) = (1 - s)^{-1}$  is unstable because it has a singularity at the pole  $s = 1$  which is in the right hand side of the complex plane.  $G^T(-s)$  is known as the adjoint of  $G(s)$  which is discussed below.

### 1.2.14 Adjoint Systems

An important concept in the ensuing development of filters and smoothers is the adjoint of a system. Consider again a system  $\mathcal{G}$  that operates on an input signal  $w$  and produces an output signal  $\mathcal{G}w$ . Then  $\mathcal{G}^H$ , the adjoint of  $\mathcal{G}$ , is the unique linear system which produces an output signal  $\mathcal{G}^H u$ , such that  $\langle u, \mathcal{G}w \rangle = \langle \mathcal{G}^H u, w \rangle$ , for all real-valued  $u$  and  $w$  of compatible dimensions. The following derivation is a simplification of the time-varying version that appears in [11].

**Lemma 1 (State-space representation of an adjoint system):** Suppose that a continuous-time linear time-invariant system  $\mathcal{G}$  is described by

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"If you thought that science was certain—well, that is just an error on your part." *Richard Phillips Feynman*

$$\dot{x}(t) = Ax(t) + Bw(t), \quad (21)$$

$$y(t) = Cx(t) + Dw(t), \quad (22)$$

with  $x(t_0) = 0$ . The adjoint  $\mathcal{G}^H$  is the linear system having the realisation

$$\dot{\zeta}(t) = -A^T \zeta(t) - C^T u(t), \quad (23)$$

$$z(t) = B^T \zeta(t) + D^T u(t), \quad (24)$$

with  $\zeta(T) = 0$ .

**Proof:** The system (21) – (22) can be written equivalently

$$\begin{bmatrix} \frac{d}{dt} I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 0(t) \\ y(t) \end{bmatrix} \quad (25)$$

with  $x(t_0) = 0$ . Thus

$$\begin{aligned} \langle y, \mathcal{G} w \rangle &= \left\langle \begin{bmatrix} \zeta \\ u \end{bmatrix}, \begin{bmatrix} \frac{d}{dt} I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle \\ &= \int_0^T \left( \zeta^T \frac{dx}{dt} \right) dt - \int_0^T \zeta^T (Ax + Bw) dt + \int_0^T u^T (Cx + Dw) dt. \end{aligned} \quad (26)$$

Integrating the last term by parts gives

$$\begin{aligned} \langle y, \mathcal{G} w \rangle &= \zeta^T(T)x(T) - \int_0^T \left( \frac{d\zeta^T}{dt} \right) x dt - \int_0^T \zeta^T (Ax + Bw) dt \\ &\quad + \int_0^T u^T (Cx + Dw) dt \\ &= \left\langle \begin{bmatrix} -\left( \frac{d}{dt} I - A^T \right) & C^T \\ -B^T & D^T \end{bmatrix} \begin{bmatrix} \zeta \\ u \end{bmatrix}, \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle + \lambda^T(T)x(T) \\ &= \langle \mathcal{G}^H y, w \rangle, \end{aligned} \quad (27)$$

where  $\mathcal{G}^H$  is given by (23) – (24).

Thus, the adjoint of a continuous-time system having the parameters  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is a system with  $\begin{bmatrix} -A^T & -C^T \\ B^T & D^T \end{bmatrix}$ . Adjoint systems have the property  $(\mathcal{G}^H)^H = \mathcal{G}$ .

The adjoint of the transfer function matrix  $G(s)$  is denoted as  $G^H(s)$  and is defined by the transfer function matrix

$$G^H(s) = G^T(-s). \quad (28)$$

**Example 7.** Suppose that a system  $\mathcal{G}$  has state-space parameters  $A = -1$  and  $B = C = D = 1$ . From (23) – (24), an adjoint system has the state-space parameters  $A = 1$ ,  $B = D = 1$  and  $C = -1$  and the corresponding transfer function is  $G^H(s) = 1 - (s - 1)^{-1} = (-s + 2)(-s + 1)^{-1} = (s - 2)(s - 1)^{-1}$ , which is unstable and non-minimum-phase. Alternatively, the adjoint of  $G(s) = 1 + (s + 1)^{-1} = (s + 2)(s + 1)^{-1}$  can be obtained using (28), namely  $G^H(s) = G^T(-s) = (-s + 2)(-s + 1)^{-1}$ .

### 1.2.15 Causal and Noncausal Systems

A causal system is a system that depends exclusively on past and current inputs.

**Example 8.** The differential of  $x(t)$  with respect to  $t$  is defined by  $\dot{x}(t) = \lim_{dt \rightarrow 0} \frac{x(t+dt) - x(t)}{dt}$ . Consider

$$\dot{x}(t) = Ax(t) + Bw(t) \quad (29)$$

with  $\text{Re}\{\lambda_i(A)\} < 0$ ,  $i = 1, \dots, n$ . The positive sign of  $\dot{x}(t)$  within (29) denotes a system that proceeds forward in time. This is called a causal system because it depends only on past and current inputs.

**Example 9.** The negative differential of  $\zeta(t)$  with respect to  $t$  is defined by  $-\dot{\zeta}(t) = \lim_{dt \rightarrow 0} \frac{\zeta(t) - \zeta(t+dt)}{dt}$ . Consider

$$-\dot{\zeta}(t) = A^T \zeta(t) + C^T u(t) \quad (30)$$

with  $\text{Re}\{\lambda_i(A)\} = \text{Re}\{\lambda_i(A^T)\} < 0$ ,  $i = 1 \dots n$ . The negative sign of  $\dot{\zeta}(t)$  within (30) denotes a system that proceeds backwards in time. Since this system depends on future inputs, it is termed noncausal. Note that  $\text{Re}\{\lambda_i(A)\} < 0$  implies

---

“Science is simply common sense at its best that is, rigidly accurate in observation, and merciless to fallacy in logic. *Thomas Henry Huxley*”

$\text{Re}\{\lambda_i(-A)\} > 0$ . Hence, if causal system (21) – (22) is stable, then its adjoint (23) – (24) is unstable.

### 1.2.16 Realising Unstable System Components

Unstable systems are termed unrealisable because their outputs are not in  $\mathcal{L}_2$  that is, they are unbounded. In other words, they cannot be implemented as forward-going systems. It follows from the above discussion that an unstable system component can be realised as a stable noncausal or backwards system.

Suppose that the time domain system  $\mathcal{G}$  is stable. The adjoint system  $z = \mathcal{G}^H u$  can be realised by the following three-step procedure.

- (i) Time-reverse the input signal  $u(t)$ , that is, construct  $u(\tau)$ , where  $\tau = T - t$  is a time-to-go variable (see [12]).
- (ii) Realise the stable system  $\mathcal{G}^T$

$$\dot{\zeta}(\tau) = A^T \zeta(\tau) + C^T u(\tau), \quad (31)$$

$$z(\tau) = B^T \zeta(\tau) + D^T u(\tau), \quad (32)$$

with  $\zeta(T) = 0$ .

- (iii) Time-reverse the output signal  $z(\tau)$ , that is, construct  $z(t)$ .

The above procedure is known as noncausal filtering or smoothing; see the discrete-time examples described in [13]. Thus, a combination of causal and noncausal system components can be used to implement an otherwise unrealisable system. This approach will be exploited in the realisation of smoothers within subsequent chapters.

**Example 10.** Suppose that it is required to realise the unstable system  $G(s) = G_2^H(s)G_1(s)$  over an interval  $[0, T]$ , where  $G_1(s) = (s+1)^{-1}$  and  $G_2(s) = (s+2)^{-1}$ . This system can be realised using the processes shown in Fig. 2.

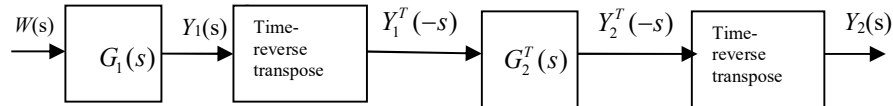


Fig. 2. Realising an unstable  $G(s) = G_2^H(s)G_1(s)$ .

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“Time is what prevents everything from happening at once.” *John Archibald Wheeler*

### 1.2.17 Power Spectral Density

The power of a voltage signal applied to a 1-ohm load is defined as the squared value of the signal and is expressed in watts. The power spectral density is expressed as power per unit bandwidth, that is, W/Hz. Consider again a linear, time-invariant system  $y = \mathcal{G}w$  and its corresponding transfer function matrix  $G(s)$ . Assume that  $w$  is a zero-mean, stationary, white noise process with  $E\{w(t)w^T(\tau)\} = Q\delta(t-\tau)$ , in which  $\delta$  denotes the Dirac delta function. Then  $\Phi_{yy}(s)$ , the power spectral density of  $y$ , is given by

$$\Phi_{yy}(s) = GQG^H(s), \quad (33)$$

which has the property  $\Phi_{yy}(s) = \Phi_{yy}(-s)$ .

The total energy of a signal is the integral of the power of the signal over time and is expressed in watt-seconds or joules. From Parseval's theorem (9), the average total energy of  $y(t)$  is

$$\int_{-\infty}^{\infty} \Phi_{yy}(s) ds = \int_{-\infty}^{\infty} |y(t)|^2 dt = \|y(t)\|_2^2 = E\{y^T(t)y(t)\}, \quad (34)$$

which is equal to the area under the power spectral density curve.

### 1.2.18 Spectral Factorisation

Suppose that noisy measurements

$$z(t) = y(t) + v(t) \quad (35)$$

of a linear, time-invariant system  $\mathcal{G}$ , described by (21) - (22), are available, where  $v(t) \in \mathbb{R}^q$  is an independent, zero-mean, stationary white noise process with  $E\{v(t)v^T(\tau)\} = R\delta(t-\tau)$ . Let

$$\Phi_{zz}(s) = GQG^H(s) + R \quad (36)$$

denote the spectral density matrix of the measurements  $z(t)$ . Spectral factorisation was pioneered by Wiener (see [4] and [5]). It refers to the problem of decomposing a spectral density matrix into a product of a stable, minimum-phase matrix transfer function and its adjoint. In the case of the output power spectral density (36), a spectral factor  $\Delta(s)$  satisfies  $\Delta(s)\Delta^H(s) = \Phi_{zz}(s)$ .

---

"Science may be described as the art of systematic over-simplification." *Karl Raimund Popper*

The problem of spectral factorisation within continuous-time Wiener filtering problems is studied in [14]. The roots of the transfer function polynomials need to be sorted into those within the left-hand-plane and the right-hand plane. This is an eigenvalue decomposition problem – see the survey of spectral factorisation methods detailed in [15].

**Example 11.** In respect of the observation spectral density (36), suppose that  $G(s) = (s + 1)^{-1}$  and  $Q = R = 1$ , which results in  $\Phi_{zz}(s) = (-s^2 + 2)(-s^2 + 1)^{-1}$ . By inspection, the spectral factor  $\Delta(s) = (s + \sqrt{2})(s + 1)^{-1}$  is stable, minimum-phase and satisfies  $\Delta(s)\Delta^H(s) = \Phi_{zz}(s)$ .

## 1.3 Minimum-Mean-Square-Error Filtering

### 1.3.1 Filter Derivation

Now that some underlying frequency-domain concepts have been introduced, the Wiener filter [4] – [6] can be described. A Wiener-Hopf derivation of the Wiener filter appears in [4], [6]. This section describes a simpler completing-the-square approach (see [14], [16]). Consider a linear time-invariant system having a transfer function matrix  $G_2(s) = C_2(sI - A)^{-1}B + D_2$ . Let  $Y_2(s)$ ,  $W(s)$ ,  $V(s)$  and  $Z(s)$  denote the Laplace transforms of the system's output, process noise, measurement noise and observations, respectively, so that

$$Z(s) = Y_2(s) + V(s). \quad (37)$$

Consider also a fictitious reference system having the transfer function  $G_1(s) = C_1(sI - A)^{-1}B + D_1$  as shown in Fig. 3. The problem is to design a filter transfer function  $H(s)$  to calculate estimates  $\hat{Y}_1(s) = H(s)Z(s)$  of  $Y_1(s)$  so that the energy  $\int_{-\infty}^{\infty} \tilde{Y}(s)\tilde{Y}^H(s)ds$  of the estimation error

$$\tilde{Y}(s) = Y_1(s) - \hat{Y}_1(s) \quad (38)$$

is minimised. It follows from the configuration of Fig. 3 that  $\tilde{Y}(s)$  is generated by

$$\tilde{Y}(s) = -[H(s) \quad HG_2(s) - G_1(s)] \begin{bmatrix} V(s) \\ W(s) \end{bmatrix}. \quad (39)$$

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“Science is what you know. Philosophy is what you don't know.” *Earl Bertrand Arthur William Russell*



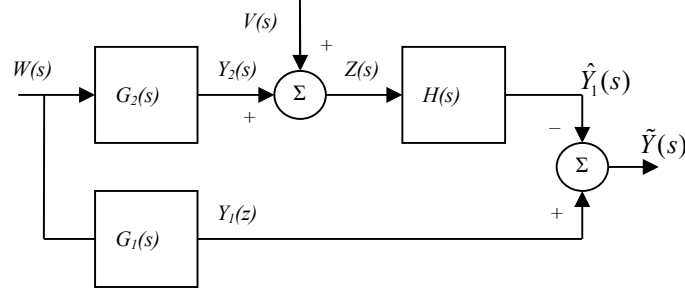


Fig. 3. The s-domain general filtering problem.

The error power spectrum density matrix is denoted by  $\Phi_{ee}(s)$  and given by the covariance of  $\tilde{Y}(s)$ , that is,

$$\begin{aligned} \Phi_{\tilde{Y}\tilde{Y}}(s) &= \tilde{Y}(s)\tilde{Y}^H(s) \\ &= \begin{bmatrix} H(s) & HG_2(s) - G_1(s) \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} H^H(s) \\ G_2^H H^H(s) - G_1^H(s) \end{bmatrix} \end{aligned} \quad (40)$$

$$= G_1 Q G_1^H(s) - G_1 Q G_2^H H^H(s) - H G_2 Q G_1^H(s) + H \Delta \Delta^H H^H(s),$$

where

$$\Delta \Delta^H(s) = G_2 Q G_2^H(s) + R \quad (41)$$

is the spectral density matrix of the measurements. The quantity  $\Delta(s)$  is called a spectral factor, which is unique up to the product of an inner matrix. Denote  $\Delta^{-H}(s) = (\Delta^H(s))^{-1}$ . Completing the square within (40) yields

$$\begin{aligned} \Phi_{ee}(s) &= G_1 Q G_1^H(s) - G_1 Q G_2^H (\Delta \Delta^H)^{-1} G_2 Q G_1^H(s) \\ &\quad + (H \Delta(s) - G_1 Q G_2^H \Delta^{-H}(s))(H \Delta(s) - G_1 Q G_2^H \Delta^{-H}(s))^H. \end{aligned} \quad (42)$$

It follows that the total energy of the error signal is given by

$$\begin{aligned} \int_{-j\infty}^{j\infty} \Phi_{\tilde{Y}\tilde{Y}}(s) ds &= \int_{-j\infty}^{j\infty} G_1 Q G_1^H(s) - G_1 Q G_2^H (\Delta \Delta^H)^{-1} G_2 Q G_1^H(s) ds \\ &\quad + \int_{-j\infty}^{j\infty} (H \Delta(s) - G_1 Q G_2^H \Delta^{-H}(s))(H \Delta(s) - G_1 Q G_2^H \Delta^{-H}(s))^H ds. \end{aligned} \quad (43)$$

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“There is an astonishing imagination, even in the science of mathematics.” *Francois-Marie Arouet de Voltaire*

The first term on the right-hand-side of (43) is independent of  $H(s)$  and represents a lower bound of  $\int_{-j\infty}^{j\infty} \Phi_{\tilde{Y}\tilde{Y}}(s)ds$ . The second term on the right-hand-side of (43) may be minimised by a judicious choice for  $H(s)$ .

**Theorem 2:** *The above linear time-invariant filtering problem with by the measurements (37) and estimation error (38) has the solution*

$$H(s) = G_1 Q G_2^H \Delta^{-H} \Delta^{-1}(s). \quad (44)$$

which minimises  $\int_{-j\infty}^{j\infty} \Phi_{\tilde{Y}\tilde{Y}}(s)ds$ .

**Proof:** *The result follows by setting  $H\Delta - G_1 Q G_2^H \Delta^{-H}(s) = 0$  within (43).*

By Parseval's theorem, the minimum mean-square-error solution (44) also minimises  $\|e(t)\|_2^2$ .

The solution (44) is unstable because the factor  $G_2^H (\Delta^H)^{-1}(s)$  possesses right-hand-plane poles. This optimal noncausal solution is actually a smoother, which can be realised by a combination of forward and backward processes. Wiener called (44) the optimal unrealisable solution because it cannot be realised by a memory-less network of capacitors, inductors and resistors [4].

The transfer function matrix of a realisable filter is given by

$$H(s) = \{G_1 Q G_2^H (\Delta^H)^{-1}\}_+ \Delta^{-1}(s), \quad (45)$$

in which  $\{\}_+$  denotes the causal part. A procedure for finding the causal part of a transfer function is described below.

### 1.3.2 Finding the Causal Part of a Transfer Function

The causal part of transfer function can be found by carrying out the following three steps.

- (i) If the transfer function is not strictly proper, that is, if the order of the numerator is not less than the degree of the denominator, then perform synthetic division to isolate the constant term.
- (ii) Expand out the (strictly proper) transfer function into the sum of stable and unstable partial fractions.
- (iii) The causal part is the sum of the constant term and the stable partial fractions.

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“The important thing in science is not so much to obtain new facts as to discover new ways of thinking about them.” *William Henry Bragg*

Incidentally, the noncausal part is what remains, namely the sum of the unstable partial fractions.

**Example 12.** Consider  $G(s) = (s^2 - \beta^2)(s^2 - \alpha^2)^{-1}$  with  $\alpha, \beta < 0$ . Since  $G_2(s)$  possesses equal order numerator and denominator polynomials, synthetic division is required, which yields  $G_2(s) = 1 + (\alpha^2 - \beta^2)(s^2 - \alpha^2)^{-1}$ . A partial fraction expansion results in

$$\frac{(\alpha^2 - \beta^2)}{(s^2 - \alpha^2)} = \frac{0.5\alpha^{-1}(\alpha^2 - \beta^2)}{(s + \alpha)} - \frac{0.5\alpha^{-1}(\alpha^2 - \beta^2)}{(s - \alpha)}.$$

Thus, the causal part of  $G(s)$  is  $\{G(s)\}_+ = 1 - 0.5\alpha^{-1}(\alpha^2 - \beta^2)(s - \alpha)^{-1}$ . The noncausal part of  $G(s)$  is denoted as  $\{G(s)\}_-$  and is given by  $\{G(s)\}_- = 0.5\alpha^{-1}(\alpha^2 - \beta^2)(s - \alpha)^{-1}$ . It is easily verified that  $G(s) = \{G(s)\}_+ + \{G(s)\}_-$ .

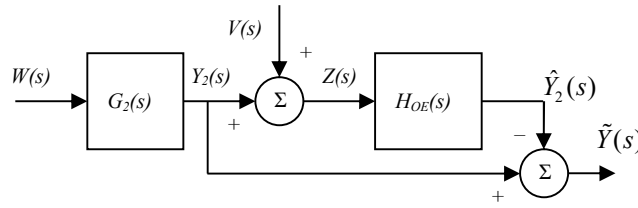


Fig. 4. The s-domain output estimation problem.

### 1.3.3 Minimum-Mean-Square-Error Output Estimation

In output estimation, the reference system is the same as the generating system, as depicted in Fig. 4. The simplification of the optimal noncausal solution (44) of Theorem 2 for the case  $G_1(s) = G_2(s)$  can be expressed as

$$\begin{aligned} H_{OE}(s) &= G_2 Q G_2^H \Delta^{-H} \Delta^{-1}(s) \\ &= G_2 Q G_2^H (\Delta \Delta^H)^{-1}(s) \\ &= (\Delta \Delta^H - R)(\Delta \Delta^H)^{-1}(s) \\ &= I - R \Delta^{-H} \Delta^{-1}(s). \end{aligned} \tag{46}$$

The optimal causal solution for output estimation is [14]

$$H_{OE}(s) = \left\{ G_2 Q G_2^H \Delta^{-H} \right\}_+ \Delta^{-1}(s)$$

$$\begin{aligned} &= I - R \left\{ \Delta^{-H} \right\}_+ \Delta^{-1}(s) \\ &= I - R^{1/2} \Delta^{-1}(s). \end{aligned} \quad (47)$$

When the measurement noise becomes negligibly small, the output estimator approaches a short circuit, that is,

$$\lim_{R \rightarrow 0, s \rightarrow 0} |H_{OE}(s)| = I. \quad (48)$$

The observation (48) can be verified by substituting  $\Delta \Delta^H(s) = G_2 Q G_2^H(s)$  into (46). This observation is consistent with intuition, that is, when the measurements are perfect, filtering will be superfluous.

**Example 13.** Consider a scalar output estimation problem, where  $G_2(s) = (s - \alpha)^{-1}$ ,  $\alpha = -1$ ,  $Q = 1$  and  $R = 0.0001$ . Then  $G_2 Q G_2^H(s) = Q(-s^2 + \alpha^2)^{-1}$  and  $\Delta \Delta^H(s) = -(Rs^2 + R\alpha^2 + Q)(-s^2 + \alpha^2)^{-1}$ , which leads to  $\Delta(s) = R^{1/2}(s + \sqrt{\alpha^2 + Q/R})(s - \alpha)^{-1}$ . Therefore,  $G_2 Q G_2^H (\Delta^H)^{-1}(s) = \frac{Q}{(s - \alpha)(-s - \alpha)} \frac{(-s - \alpha)}{R^{1/2}(-s + \sqrt{\alpha^2 + Q/R})} = \frac{Q}{R^{1/2}(s - \alpha)(-s + \sqrt{\alpha^2 + Q/R})}$ , in which a common pole and zero were cancelled. Expanding into partial fractions and taking the causal part results in

$$\{G_2 Q G_2^H (\Delta^H)^{-1}(s)\}_+ = \frac{\left. \frac{Q}{R^{1/2}(-s + \sqrt{\alpha^2 + Q/R})} \right|_{s=\alpha}}{(s - \alpha)}$$

and

$$H_{OE}(s) = \{G_2 Q G_2^H \Delta^{-H}\}_+ \Delta^{-1}(s) = \frac{\alpha + \sqrt{\alpha^2 + Q/R}}{s + \sqrt{\alpha^2 + Q/R}}.$$

Substituting  $\alpha = -1$ ,  $Q = 1$  and  $R = 0.0001$  yields  $H(s) = 99(s + 100)^{-1}$ . By

inspection,  $\lim_{s \rightarrow 0} |H(s)| = \frac{99}{100}$ , which illustrates the low measurement noise

asymptote (48). Some sample trajectories from a simulation conducted with  $\delta_t = 0.001$  s are shown in Fig. 5. The input measurements are shown in Fig. 5(a). It can be seen that the filtered signal (the solid line of Fig. 5 (b)) estimates the system output (the dotted line of Fig. 5(b)).

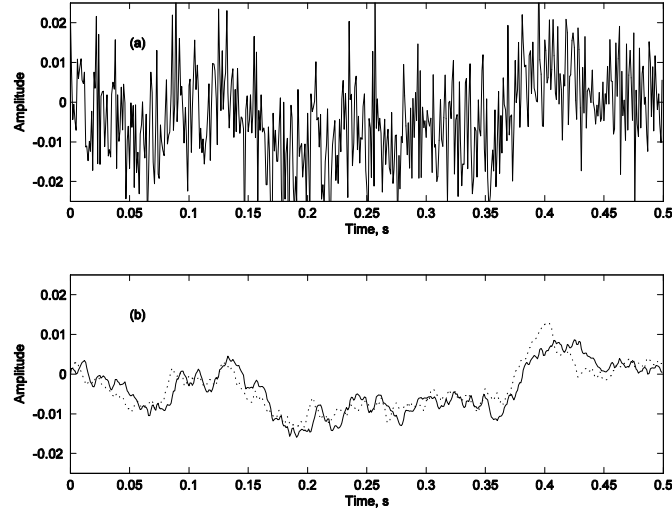


Fig. 5. Sample trajectories for Example 13: (a) measurement, (b) system output (dotted line) and filtered signal (solid line).

### 1.3.4 Minimum-Mean-Square-Error Input Estimation

In input estimation problems, it is desired to estimate the input process  $w(t)$ , as depicted in Fig. 6. This is commonly known as an equalisation problem, in which it is desired to mitigate the distortion introduced by a communication channel  $G_2(s)$ . The simplification of the general noncausal solution (44) of Theorem 2 for the case of  $G_2(s) = I$  results in

$$H_{IE}(s) = Q G_2^H \Delta^{-H} \Delta^{-1}(s). \quad (49)$$

Equation (49) is known as the optimum minimum-mean-square-error noncausal equaliser [17]. Assume that:  $G_2(s)$  is proper, that is, the order of the numerator is the same as the order of the denominator, and the zeros of  $G_2(s)$  are in the left-hand-plane. Under these conditions, when the measurement noise becomes negligibly small, the equaliser estimates the inverse of the system model, that is,

$$\lim_{R \rightarrow 0} H_{IE}(s) = G_2^{-1}(s). \quad (50)$$

The observation (50) can be verified by substituting  $\Delta \Delta^H(s) = G_2 Q G_2^H(s)$  into (49). In other words, if the channel model is invertible and signal to noise ratio is sufficiently high, the equaliser will estimate  $w(t)$ . When measurement noise is

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“All of the biggest technological inventions created by man - the airplane, the automobile, the computer - says little about his intelligence, but speaks volumes about his laziness.” *Mark Raymond Kennedy*

present the equaliser no longer approximates the channel inverse because some filtering is also required. In the limit, when the signal to noise ratio is sufficiently low, the equaliser approaches an open circuit, namely,

$$\lim_{Q \rightarrow 0, s \rightarrow 0} |H_{IE}(s)| = 0. \quad (51)$$

The observation (51) can be verified by substituting  $Q = 0$  into (49). Thus, when the equalisation problem is dominated by measurement noise, the estimation error is minimised by ignoring the data.

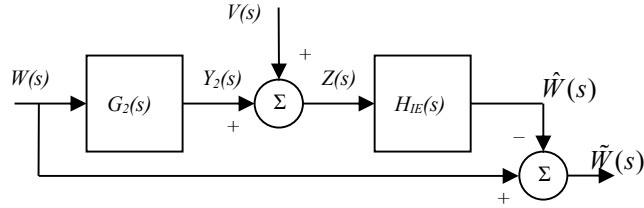


Fig. 6. The s-domain input estimation problem.

## 1.4 Chapter Summary

Continuous-time, linear, time-invariant systems can be described via either a differential equation model or as a state-space model. Signal models can be written in the time-domain as

$$\begin{aligned} & \left( a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \frac{d}{dt} + a_0 \right) y(t) \\ &= \left( b_m \frac{d^m}{dt^m} + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} + \dots + b_1 \frac{d}{dt} + b_0 \right) w(t). \end{aligned}$$

Under the time-invariance assumption, the system transfer function matrices exist, which are written as polynomial fractions in the Laplace transform variable

$$Y(s) = \left[ \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \right] W(s) = G(s)W(s).$$

Thus, knowledge of a system's differential equation is sufficient to identify its transfer function. If the poles of a system's transfer function are all in the left-hand-plane then the system is asymptotically stable. That is, if the input to the system is bounded then the output of the system will be bounded.

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“Read Euler, read Euler, he is our master in everything.” *Pierre-Simon Laplace*

The optimal solution minimises the energy of the error in the time domain. It is found in the frequency domain by minimising the mean-square-error. The main results are summarised in Table 1. The optimal noncausal solution has unstable factors. It can only be realised by a combination of forward and backward processes, which is known as smoothing. The optimal causal solution is also known as the Wiener filter.

	ASSUMPTIONS	MAIN RESULTS
Signals and systems	$E\{w(t)\} = E\{W(s)\} = E\{v(t)\} = E\{V(s)\} = 0$ . $E\{w(t)w^T(t)\} = E\{W(s)W^T(s)\} = Q > 0$ and $E\{v(t)v^T(t)\} = E\{V(s)V^T(s)\} = R > 0$ are known. $A$ , $B$ , $C_1$ , $C_2$ , $D_1$ and $D_2$ are known. $G_1(s)$ and $G_2(s)$ are stable, i.e., $\text{Re}\{\lambda_i(A)\} < 0$ .	$G_1(s) = C_1(sI - A)^{-1}B + D_1$ $G_2(s) = C_2(sI - A)^{-1}B + D_2$
Spectral factorisation	$\Delta(s)$ and $\Delta^{-1}(s)$ are stable, i.e., the poles and zeros of $\Delta(s)$ are in the left-half-plane.	$\Delta\Delta^H(s) = G_2QG_2^H(s) + R$
Non-causal solution		$H(s) = G_1QG_2^H(\Delta^H)^{-1}\Delta^{-1}(s)$
Causal solution		$H(s) = \{G_1QG_2^H(\Delta^H)^{-1}\}_+ \Delta^{-1}(s)$

Table 1. Main results for the continuous-time general filtering problem.

In output estimation problems,  $C_1 = C_2$ ,  $D_1 = D_2$ , that is,  $G_1(s) = G_2(s)$  and when the measurement noise becomes negligible, the solution approaches a short circuit. In input estimation or equalisation,  $C_1 = 0$ ,  $D_1 = I$ , that is,  $G_1(s) = I$  and when the measurement noise becomes negligible, the optimal equaliser approaches the channel inverse, provided the inverse exists. Conversely, when the problem is dominated by measurement noise then the equaliser approaches an open circuit.

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“Madam, I have come from a country where people are hanged if they talk.” *Leonard Euler*

## 1.5 Problems

**Problem 1.** Find the transfer functions and comment on stability of the systems having the following polynomial fractions.

- (a)  $\ddot{y} + 7\dot{y} + 12y = \ddot{w} + \dot{w} - 2w$ .
- (b)  $\ddot{y} + 1\dot{y} - 20y = \ddot{w} + 5\dot{w} + 6w$ .
- (c)  $\ddot{y} + 11\dot{y} + 30y = \ddot{w} - 7\dot{w} + 12w$ .
- (d)  $\ddot{y} - 13\dot{y} + 42y = \ddot{w} + 9\dot{w} + 20w$ .
- (e)  $\ddot{y} - 15\dot{y} + 56y = \ddot{w} + 11\dot{w} + 30w$ .

**Problem 2.** Find the transfer functions and comment on the stability for systems having the following state-space parameters.

- (a)  $A = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $C = [-6 \quad -14]$  and  $D = 1$ .
- (b)  $A = \begin{bmatrix} -7 & 20 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $C = [-2 \quad 26]$  and  $D = 1$ .
- (c)  $A = \begin{bmatrix} -11 & -30 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $C = [-18 \quad -18]$  and  $D = 1$ .
- (d)  $A = \begin{bmatrix} 13 & -42 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $C = [22 \quad -22]$  and  $D = 1$ .
- (e)  $A = \begin{bmatrix} -15 & -56 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $C = [-4 \quad -26]$  and  $D = 1$ .

**Problem 3.** Calculate the spectral factors for  $\Phi_{\underline{z}}(s) = GQG^H(s) + R$  having the following models and noise statistics.

- (a)  $G(s) = (s+1)^{-1}$ ,  $Q = 2$  and  $R = 1$ .
- (b)  $G(s) = (s+2)^{-1}$ ,  $Q = 5$  and  $R = 1$ .
- (c)  $G(s) = (s+3)^{-1}$ ,  $Q = 7$  and  $R = 1$ .
- (d)  $G(s) = (s+4)^{-1}$ ,  $Q = 9$  and  $R = 1$ .
- (e)  $G(s) = (s+5)^{-1}$ ,  $Q = 11$  and  $R = 1$ .

**Problem 4.** Calculate the optimal causal output estimators for Problem 3.

**Problem 5.** Consider the error spectral density matrix

$$\Phi_{\bar{Y}\bar{Y}}(s) = [H\Delta - G_1QG_2^H(\Delta^H)^{-1}][H\Delta - G_1QG_2^H(\Delta^H)^{-1}]^H(s) \\ + [G_1QG_1^H - G_1QG_2^H(\Delta\Delta^H)^{-1}G_2QG_1^H](s).$$

- (a) Derive the optimal output estimator.
- (b) Derive the optimal causal output estimator.

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“Nothing shocks me. I’m a scientist.” *Harrison Ford*



(c) Derive the optimal input estimator.

**Problem 6** [16]. In respect of the configuration in Fig. 2, suppose that

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 25 \\ 25 \\ -25 \end{bmatrix}, \quad C_2 = [-1 \quad 2 \quad 1], \quad C_1 = [1 \quad 1 \quad 1], \quad D = 0, \quad Q = 1$$

and  $R = 1$ . Show that the optimal causal filter is given by  $H(s) = -(16.9s^2 + 86.5s + 97.3)(s^3 + 8.64s^2 + 30.3s + 50.3)^{-1}$ .

**Problem 7** [18]. Suppose that  $G_2 Q G_2^H(s) = \frac{-3600}{s^2(169 - s^2)}$  and  $R(s) = 1$ . Show that

the optimal causal filter for output estimation is given by  $H_{OE}(s) = (4s + 60)(s^2 + 17s + 60)^{-1}$ .

## 1.6 Glossary

The following terms have been introduced within this section.

$\mathbb{R}$	The space of real numbers.
$\mathbb{R}^n$	The space of real-valued $n$ -element column vectors.
$t$	The real-valued continuous-time variable. For example, $t \in (-\infty, \infty)$ and $t \in [0, \infty)$ denote $-\infty < t < \infty$ and $0 \leq t < \infty$ , respectively.
$w(t) \in \mathbb{R}^n$	A continuous-time, real-valued, $n$ -element stationary stochastic input signal.
$w$	The matrix of $w(t)$ all time, i.e., $w = [w(-\infty), \dots, w(\infty)]$ .
$y = \mathcal{G}w$	The output of a linear system $\mathcal{G}$ that operates on an input signal $w$ .
$A, B, C, D$	Time-invariant state space matrices of appropriate dimension. The system $\mathcal{G}$ is assumed to have the realisation $\dot{x}(t) = Ax(t) + Bw(t)$ , $y(t) = Cx(t) + Dw(t)$ in which $w(t)$ is known as the process noise or input signal.
$v(t)$	A stationary stochastic measurement noise signal.
$\delta(t)$	The Dirac delta function.
$Q$ and $R$	Time-invariant covariance matrices of stochastic signals $w(t)$ and $v(t)$ , respectively.
$s$	The Laplace transform variable.
$Y(s)$	The Laplace transform of a continuous-time signal $y(t)$ .
$G(s)$	The transfer function matrix of a system $\mathcal{G}$ . For example, the transfer function matrix of the system $\dot{x}(t) = Ax(t) + Bw(t)$ , $y(t) = Cx(t) + Dw(t)$ is given by $G(s) = C(sI - A)^{-1}B + D$ .
$\langle v, w \rangle$	The inner product of two continuous-time signal vectors $v$

	and $w$ which is defined by $\langle v, w \rangle = \int_{-\infty}^{\infty} v^T w \, dt$ .
$\ w\ _2$	The 2-norm of the continuous-time signal vector $w$ which is defined by $\ w\ _2 = \sqrt{\langle w, w \rangle} = \sqrt{\int_{-\infty}^{\infty} w^T w \, dt}$ .
$\mathcal{L}_2$	The set of continuous-time signals having finite 2-norm, which is known as the Lebesgue 2-space.
$\lambda_i(A)$	The $i$ eigenvalues of $A$ .
$\text{Re}\{\lambda_i(A)\}$	The real part of the eigenvalues of $A$ .
Asymptotic stability	A linear system $\mathcal{G}$ is said to be asymptotically stable if its output $y \in \mathcal{L}_2$ for any $w \in \mathcal{L}_2$ . If $\text{Re}\{\lambda_i(A)\}$ are in the left-hand-plane or equivalently if the real part of transfer function's poles are in the left-hand-plane then the system is stable.
$\mathcal{G}^H$	The adjoint of $\mathcal{G}$ . The adjoint of a system having the state-space parameters $\{A, B, C, D\}$ is a system parameterised by $\{-A^T, -C^T, B^T, D^T\}$ .
$G^H(s)$	The adjoint (or Hermitian transpose) of the transfer function matrix $G(s)$ .
$\Phi_{zz}(s)$	The spectral density matrix of the measurements $z$ .
$\Delta(s)$	The spectral factor of $\Phi_{zz}(s)$ which satisfies $\Delta \Delta^H(s) = G Q G^H(s) + R$ and $\Delta^{-H}(s) = (\Delta^H)^{-1}(s)$ .
$G^{-1}(s)$	Inverse of the transfer function matrix $G(s)$ .
$G^{-H}(s)$	Inverse of the adjoint transfer function matrix $G^H(s)$ .
$\{G(s)\}_+$	Causal part of the transfer function matrix $G(s)$ .
$H(s)$	Transfer function matrix of the minimum mean-square-error solution.
$H_{OE}(s)$	Transfer function matrix of the minimum mean-square-error solution specialised for output estimation.
$H_{IE}(s)$	Transfer function matrix of the minimum mean-square-error solution specialised for input estimation.

## 1.7 References

- [1] O. Neugebauer, *A history of ancient mathematical astronomy*, Springer, Berlin and New York, 1975.
- [2] C. F. Gauss, *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientum*, Hamburg, 1809 (Translated: *Theory of the Motion of the Heavenly Bodies*, Dover, New York, 1963).
- [3] A. N. Kolmogorov, "Sur l'interpolation et extrapolation des suites stationnaires", *Comptes Rendus. de l'Academie des Sciences*, vol. 208, pp. 2043 – 2045, 1939.
- [4] N. Wiener, *Extrapolation, interpolation and smoothing of stationary time series with engineering applications*, The MIT Press, Cambridge Mass.; Wiley, New York; Chapman &

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"Facts are stupid things." *Ronald Wilson Reagan*

- Hall, London, 1949.
- [5] P. Masani, “Wiener’s Contributions to Generalized Harmonic Analysis, Prediction Theory and Filter Theory”, *Bulletin of the American Mathematical Society*, vol. 72, no. 1, pt. 2, pp. 73 – 125, 1966.
  - [6] T. Kailath, *Lectures on Wiener and Kalman Filtering*, Springer Verlag, Wien; New York, 1981.
  - [7] M.-A. Parseval Des Chênes, *Mémoires présentés à l’Institut des Sciences, Lettres et Arts, par divers savans, et lus dans ses assemblées. Sciences mathématiques et physiques (Savans étrangers)*, vol. 1, pp. 638 – 648, 1806.
  - [8] C. A. Desoer and M. Vidyasagar, *Feedback Systems : Input Output Properties*, Academic Press, N.Y., 1975.
  - [9] G. A. Einicke, “Asymptotic Optimality of the Minimum-Variance Fixed-Interval Smoother”, *IEEE Transactions on Signal Processing*, vol. 55, no. 4, pp. 1543 – 1547, Apr. 2007.
  - [10] T. Kailath, *Linear Systems*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1980.
  - [11] D. J. N. Limebeer, B. D. O. Anderson, P. Khargonekar and M. Green, “A Game Theoretic Approach to  $H_\infty$  Control for Time-varying Systems”, *SIAM Journal of Control and Optimization*, vol. 30, no. 2, pp. 62 – 283, 1992.
  - [12] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1995.
  - [13] C. S. Burrus, J. H. McClellan, A. V. Oppenheim, T. W. Parks, R. W. Schafer and H. W. Schuessler, *Computer-Based Exercises for Signal Processing Using Matlab*, Prentice-Hall, Englewood Cliffs, New Jersey, 1994.
  - [14] U. Shaked, “A general transfer function approach to linear stationary filtering and steady state optimal control problems”, *International Journal of Control*, vol. 24, no. 6, pp. 741 – 770, 1976.
  - [15] A. H. Sayed and T. Kailath, “A Survey of Spectral Factorization Methods”, *Numerical Linear Algebra with Applications*, vol. 8, pp. 467 – 496, 2001.
  - [16] U. Shaked, “ $H_\infty$ -Minimum Error State Estimation of Linear Stationary Processes”, *IEEE Transactions on Automatic Control*, vol. 35, no. 5, pp. 554 – 558, May 1990.
  - [17] S. A. Kassam and H. V. Poor, “Robust Techniques for Signal Processing: A Survey”, *Proceedings of the IEEE*, vol. 73, no. 3, pp. 433 – 481, Mar. 1985.
  - [18] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York, 1971.

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“All science is either physics or stamp collecting.” *Baron William Thomson Kelvin*

## 2. Discrete-Time, Minimum-Mean-Square-Error Filtering

### 2.1 Introduction

This chapter reviews the solutions for the discrete-time, linear stationary filtering problems that are attributed to Wiener [1] and Kolmogorov [2]. As in the continuous-time case, a model-based approach is employed. Here, a linear model is specified by the coefficients of the input and output difference equations. It is shown that the same coefficients appear in the system's (frequency domain) transfer function. In other words, frequency domain model representations can be written down without background knowledge of z-transforms.

In the 1960s and 1970s, continuous-time filters were implemented on analogue computers. This technology has been largely discontinued for two main reasons. First, analogue multipliers and op amp circuits exhibit poor performance whenever (temperature-sensitive) calibrations become out of date. Second, updated software releases are faster to turn around than hardware design iterations. Continuous-time filters are now routinely implemented using digital computers, provided that the signal sampling rates and data processing rates are sufficiently high. Alternatively, continuous-time model parameters may be converted into discrete-time and differential equations can be transformed into difference equations. The ensuing discrete-time filter solutions are then amenable to more economical implementation, namely, employing relatively lower processing rates.

The discrete-time Wiener filtering problem is solved in the frequency domain. Once again, it is shown that the optimum minimum-mean-square-error solution is found by completing the square. The optimum solution is noncausal, which can only be implemented by forward and backward processes. This solution is actually a smoother and the optimum filter is found by taking the causal part.

The developments rely on solving a spectral factorisation problem, which requires pole-zero cancellations. Therefore, some pertinent discrete-time concepts are introduced in Section 2.2 prior to deriving the filtering results. The discussion of the prerequisite concepts is comparatively brief since it mirrors the continuous-time material introduced previously. In Section 2.3 it is shown that the structure of the filter solutions is unchanged – only the spectral factors are calculated differently.

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“If we value the pursuit of knowledge, we must be free to follow wherever that search may lead us. The free mind is not a barking dog, to be tethered on a ten foot-chain.” *Adlai Ewing Stevenson Jr.*

## 2.2 Prerequisites

### 2.2.1 Spaces

A discrete-time, real-valued stochastic process is denoted as  $w_k = [w_{1,k}, w_{2,k}, \dots, w_{n,k}]^T \in \mathbb{R}^n$  for integer time step  $k \in (-\infty, \infty)$ . Suppose that  $w_k, v_k$  are scalars *i.e.*,  $w_k, v_k \in \mathbb{R}$ . The set of  $w_k$  and  $v_k$  over all  $k$  are denoted by row vectors  $w = [w_{-\infty}, \dots, w_{\infty}]$ , and  $v = [v_{-\infty}, \dots, v_{\infty}]$ . The inner product  $\langle v, w \rangle$  of two discrete-time vectors  $v$  and  $w$  is defined by

$$\langle v, w \rangle = \sum_{k=-\infty}^{\infty} v_k^T w_k. \quad (1)$$

The 2-norm or Euclidean norm of a discrete-time vector process  $w$ , is defined as

$$\|w\|_2 = \sqrt{\langle w, w \rangle} = \sqrt{\sum_{k=-\infty}^{\infty} w_k^T w_k}. \text{ The square of the 2-norm, that is, } \|w\|_2^2 = \langle w, w \rangle = \sum_{k=-\infty}^{\infty} w_k^T w_k \text{ is commonly known as energy of the signal } w.$$

Consider instead  $w_k, v_k \in \mathbb{R}^n, n > 1$ . Then the set of  $w_k$  and  $v_k$  over all  $k$  are denoted by matrices  $w = [w_{-\infty}, \dots, w_{\infty}]$ , and  $v = [v_{-\infty}, \dots, v_{\infty}]$ . The inner product of  $v$  and  $w$  is defined by  $\langle v, w \rangle = \text{trace}(v^T w) = \sum_{i=1}^n \sum_{k=-\infty}^{\infty} v_{i,k} w_{i,k}$ .

**Example 1.** Consider  $w_k = [w_{1,k}, w_{2,k}, \dots, w_{n,k}]^T$  over  $k \in [1, N]$  and

$$w = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,N} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n,1} & w_{n,1} & \vdots & w_{n,N} \end{bmatrix}. \text{ It is easily verified that the energy of } w \text{ is } \langle w, w \rangle = \text{trace}(w^T w) = (w_{1,1}^2 + w_{2,1}^2 + \dots + w_{n,1}^2) + (w_{1,2}^2 + w_{2,2}^2 + \dots + w_{n,2}^2) + \dots + (w_{1,N}^2 + w_{2,N}^2 + \dots + w_{n,N}^2) = \sum_{i=1}^n \sum_{k=1}^N w_{i,k}^2, \text{ which reverts to (1) for } n = 1.$$

The Lebesgue 2-space is denoted by  $\ell_2$  and is defined as the set of discrete-time processes having a finite 2-norm. Thus,  $w \in \ell_2$  means that the energy of  $w$  is bounded. See [3] for more detailed discussions of spaces and norms.

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“To live effectively is to live with adequate information.” *Norbert Wiener*

## 2.2.2 Discrete-time Polynomial Fraction Systems

Consider a linear, time-invariant system  $\mathcal{G}$  that operates on a scalar input signal  $w = [w_1, \dots, w_N] \in \mathbb{R}^N$  and produces a scalar output signal  $y = [y_1, \dots, y_N] \in \mathbb{R}^N$ , i.e.,  $y = \mathcal{G}w$ . Such a system can be realised by the difference equation

$$a_n y_{k-n} + a_{n-1} y_{k-n+1} + \dots + a_1 y_{k-1} + a_0 y_k = b_m w_{k-m} + b_{m-1} w_{k-m+1} + \dots + b_1 w_{k-1} + b_0 w_k, \quad (2)$$

where  $a_0, \dots, a_n, b_0, \dots, b_n$  are real-valued constant coefficients, with  $a_n \neq 0$  and zero initial conditions. Difference equations of the form (2) are called discrete-time polynomial systems.

**Example 2.** The difference equation  $y_k = 0.1x_k + 0.2x_{k-1} + 0.3y_{k-1}$  specifies a system in which the coefficients are  $a_0 = 1$ ,  $a_1 = -0.3$ ,  $b_0 = 0.2$  and  $b_1 = 0.3$ . Note that  $y_k$  is known as the current output and  $y_{k-1}$  is known as a past output.

## 2.2.3 The Z-Transform of a Discrete-time Sequence

The two-sided z-transform of a discrete-time process,  $y_k$ , is denoted by  $Y(z)$  and is defined by

$$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k}, \quad (3)$$

where  $z = e^{j\omega T}$  and  $j = \sqrt{-1}$ . Given a process  $y_k$  with z-transform  $Y(z)$ ,  $y_k$  can be calculated from  $Y(z)$  by taking the inverse z-transform of  $Y(z)$ ,

$$y_k = \frac{1}{2\pi j} \int_{-e^{-j\omega T}}^{e^{j\omega T}} Y(z) z^{k-1} dz. \quad (4)$$

**Theorem 1 Parseval's Theorem:**

$$\int_{-\infty}^{\infty} |y_k|^2 dk = \frac{1}{2\pi j} \int_{-e^{-j\omega T}}^{e^{j\omega T}} |Y(z)|^2 dz. \quad (5)$$

That is, the energy in the time domain equals the energy in the frequency domain.

## 2.2.4 Polynomial Fraction Transfer Functions

In the continuous-time case, a system's differential equations lead to a transfer function in the Laplace transform variable. Here, in discrete-time, a system's

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“There is no philosophy which is not founded upon knowledge of the phenomena, but to get any profit from this knowledge it is absolutely necessary to be a mathematician.” *Daniel Bernoulli*

difference equations lead to a transfer function in the  $z$ -transform variable. Applying the  $z$ -transform to both sides of (2) yields the difference equation

$$\begin{aligned} (a_n z^{-n} + a_{n-1} z^{-n+1} + \dots + a_1 z^{-1} + a_0) Y(z) \\ = (b_m z^{-m} + b_{m-1} z^{-m+1} + \dots + b_1 z^{-1} + b_0) W(z). \end{aligned} \quad (6)$$

Therefore

$$\begin{aligned} Y(z) &= \left[ \frac{b_m z^{-m} + b_{m-1} z^{-m+1} + \dots + b_1 z^{-1} + b_0}{a_n z^{-n} + a_{n-1} z^{-n+1} + \dots + a_1 z^{-1} + a_0} \right] W(z) \\ &= G(z) W(z), \end{aligned} \quad (7)$$

where

$$G(z) = \left[ \frac{b_m z^{-m} + b_{m-1} z^{-m+1} + \dots + b_1 z^{-1} + b_0}{a_n z^{-n} + a_{n-1} z^{-n+1} + \dots + a_1 z^{-1} + a_0} \right] \quad (8)$$

is known as the transfer function of the system. It can be seen that knowledge of the system differential equation (2) is sufficient to identify its transfer function (8).

### 2.2.5 Poles and Zeros

The numerator and denominator polynomials of (8) can be factored into  $m$  and  $n$  linear factors, respectively, to give

$$G(z) = \frac{b_m (z - \beta_1)(z - \beta_2) \dots (z - \beta_m)}{a_n (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n)}. \quad (9)$$

The numerator of  $G(z)$  is zero when  $z = \beta_i$ ,  $i = 1 \dots m$ . These values of  $z$  are called the zeros of  $G(z)$ . Zeros inside the unit circle are called minimum-phase whereas zeros outside the unit circle are called non-minimum phase. The denominator of  $G(z)$  is zero when  $z = \alpha_i$ ,  $i = 1 \dots n$ . These values of  $z$  are called the poles of  $G(z)$ .

**Example 3.** Consider a system described by the difference equation  $y_k + 0.3y_{k-1} + 0.04y_{k-2} = w_k + 0.5w_{k-1}$ . It follows from (2) and (8) that the corresponding transfer function is given by

$$G(z) = \frac{1 + 0.5z^{-1}}{1 + 0.3z^{-1} + 0.04z^{-2}} = \frac{z^2 + 0.5z}{z^2 + 0.3z + 0.04} = \frac{z(z + 0.5)}{(z - 0.1)(z + 0.4)}$$

which possesses poles at  $z = 0.1, -0.4$  and zeros at  $z = 0, -0.5$ .

---

“A mathematician is a blind man in a dark room looking for a black cat which isn't there.” *Charles Robert Darwin*

## 2.2.6 Polynomial Fraction Transfer Function Matrix

In the single-input-single-output case, it is assumed that  $w(z)$ ,  $G(z)$  and  $y(z) \in \mathbb{R}$ . In the multiple-input-multiple-output case,  $G(z)$  is a transfer function matrix. For example, suppose that  $w(z) \in \mathbb{R}^m$ ,  $y(z) \in \mathbb{R}^p$ , then  $G(z) \in \mathbb{R}^{p \times m}$ , namely

$$G(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) & \dots & G_{1m}(z) \\ G_{21}(z) & G_{22}(z) & & \\ \vdots & & \ddots & \vdots \\ G_{p1}(z) & & \dots & G_{pm}(z) \end{bmatrix}, \quad (10)$$

where the components  $G_{ij}(z)$  have the polynomial transfer function form within (8) or (9).

## 2.2.7 State-Space Transfer Function Matrix

The transfer function matrix (10) can be written in the state-space representation

$$G(z) = C(zI - A)^{-1}B + D, \quad (11)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$  and  $D \in \mathbb{R}^{p \times m}$ .

**Example 4.** For a state-space model with  $A = -0.5$ ,  $B = C = 1$  and  $D = 0$ , the transfer function is  $G(z) = (z^{-1} - 0.5)^{-1}$ .

**Example 5.** For state-space parameters  $A = \begin{bmatrix} -0.33 & -0.04 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,

$C = \begin{bmatrix} 0.2 & -0.04 \end{bmatrix}$  and  $D = 1$ , the use of Cramer's rule, that is,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ , yields the transfer function  $G(z) = \frac{z(z + 0.5)}{(z - 0.1)(z + 0.4)}$ .

## 2.2.8 State-Space Realisation

The state-space transfer function matrix (11) can be realised as a discrete-time system  $\mathcal{G}: \mathbb{R}^{m \times N} \rightarrow \mathbb{R}^{p \times N}$

$$x_{k+1} = Ax_k + Bw_k, \quad (12)$$

$$y_k = Cx_k + Dw_k, \quad (13)$$

---

"I do not like it, and I am sorry I ever had anything to do with it." *Erwin Rudolf Josef Alexander Schrödinger*



where  $x_k \in \mathbb{R}^n$  is a state vector. This system is depicted in Fig. 1. The  $A = \{a_{i,j}\}$ ,  $B = \{b_{i,j}\}$ ,  $C = \{c_{i,j}\}$  and  $D = \{d_{i,j}\}$  are known as a state matrix, an input mapping, an output mapping and a direct feed-through matrix, respectively. For ease of understanding, the state-space model (15) – (16) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix}_{k+1} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_k + \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}_k$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}_k = \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{p,1} & \cdots & c_{p,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_k + \begin{bmatrix} d_{1,1} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots \\ d_{p,1} & \cdots & d_{p,m} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}_k.$$

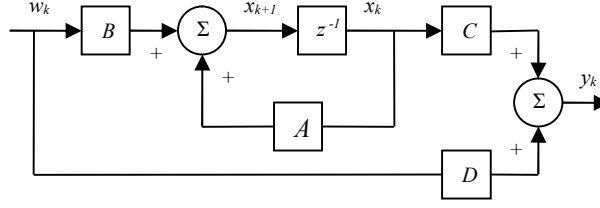


Fig. 1. Discrete-time state-space system.

It is assumed that  $w_k$  is a zero-mean, stationary process with  $E\{w_j w_k^T\} = Q$ . In most applications, discrete-time implementations are desired, however, the polynomial fraction transfer function or state-space transfer function parameters may be known in continuous-time. Therefore, two methods for transforming continuous-time parameters to discrete-time are set out below.

### 2.2.9 The Bilinear Approximation

Transfer functions in the  $z$ -plane can be mapped exactly to the  $s$ -plane by substituting  $z = e^{sT_s}$ , where  $s = j\omega$  and  $T_s$  is the sampling period. Conversely, the substitution

$$\begin{aligned} s &= \frac{1}{T_s} \log(z) \\ &= \frac{2}{T_s} \left[ \frac{z-1}{z+1} + \frac{1}{3} \left( \frac{z-1}{z+1} \right)^3 + \frac{1}{5} \left( \frac{z-1}{z+1} \right)^5 + \frac{1}{7} \left( \frac{z-1}{z+1} \right)^7 + \dots \right] \end{aligned} \quad (14)$$

---

“The beginning of knowledge is the discovery of something we do not understand.” *Frank Patrick Herber*

can be used to map s-plane transfer functions into the z-plane. The bilinear transform is a first order approximation to (14), namely,

$$s \approx \frac{2}{T_s} \left[ \frac{z-1}{z+1} \right]. \quad (15)$$

**Example 6.** Consider the continuous-time transfer function  $H(s) = (s+2)^{-1}$  with  $T_s = 2$ . Substituting (15) yields the discrete-time transfer function  $H(z) = (3z+1)^{-1}$ . The higher order terms within the series of (14) can be included to improve the accuracy of converting a continuous-time model to discrete time.

## 2.2.10 Discretisation of Continuous-time Systems

he discrete-time state-space parameters, denoted here by  $\{A_D, B_D, C_D, D_D, Q_D, R_D\}$ , can be obtained by discretising the continuous-time system

$$\dot{x}(t) = A_C x(t) + B_C w(t), \quad (16)$$

$$y(t) = C_C x(t) + D_C w(t), \quad (17)$$

$$z(t) = y(t) + v(t), \quad (18)$$

where  $E\{w(t)w^T(\tau)\} = Q_C \delta(t-\tau)$  and  $E\{v(t)v^T(\tau)\} = R_C \delta(t-\tau)$ . Premultiplying

(16) by  $e^{-A_C t}$  and recognising that  $\frac{d}{dt}(e^{-A_C t} x(t)) = e^{-A_C t} \dot{x}(t) - e^{-A_C t} A_C x(t)$  yields

$$\frac{d}{dt}(e^{-A_C t} x(t)) = e^{-A_C t} B_C w(t). \quad (19)$$

Integrating (19) results in

$$e^{-A_C t} x(t) - e^{-A_C t_0} x(t_0) = \int_{t_0}^t e^{-A_C \tau} B_C w(\tau) d\tau \quad (20)$$

and hence

$$\begin{aligned} x(t) &= e^{A_C(t-t_0)} x(t_0) + e^{A_C t} \int_{t_0}^t e^{-A_C \tau} B_C w(\tau) d\tau \\ &= e^{A_C(t-t_0)} x(t_0) + \int_{t_0}^t e^{A_C(t-\tau)} B_C w(\tau) d\tau \end{aligned} \quad (21)$$

is a solution to the differential equation (16). Suppose that  $x(t)$  is available at integer  $k$  multiples of  $T_s$ . Assuming that  $w(t)$  is constant during the sampling interval and substituting  $t_0 = kT_s$ ,  $t = (k+1)T_s$  into (21) yields

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“In the fields of observation, chance favours only the mind that is prepared.” *Louis Pasteur*

$$x((k+1)T_s) = e^{A_c T_s} x(kT_s) + \int_{kT_s}^{(k+1)T_s} e^{A_c((k+1)T_s - \tau)} B_c d\tau w(kT_s). \quad (22)$$

With the identifications  $x_k = x(kT_s)$  and  $w_k = w(kT_s)$  in (22), it can be seen that

$$A_D = e^{A_c T_s}, \quad (23)$$

$$B_D = \int_{kT_s}^{(k+1)T_s} e^{A_c((k+1)T_s - \tau)} B_c d\tau. \quad (24)$$

The  $\tau$  within the definite integral (24) varies from  $kT_s$  to  $(k+1)T_s$ . For a change of variable  $\lambda = (k+1)T_s - \tau$ , the limits of integration become  $\lambda = T_s$  and  $\lambda = 0$ , which results in the simplification

$$\begin{aligned} B_D &= - \int_{T_s}^0 e^{A_c \lambda} B_c d\lambda \\ &= \int_0^{T_s} e^{A_c \lambda} B_c d\lambda. \end{aligned} \quad (25)$$

Denoting  $E\{w_j w_k^T\} = Q_D \delta_{jk}$  and using (25) it can be shown that [4]

$$Q_D = \int_0^{T_s} e^{A_c \lambda} B_c Q_c B_c^T e^{A_c \lambda} d\lambda. \quad (26)$$

The exponential matrix is defined as

$$e^{A_c t} = I + A_c t + \frac{A_c^2 t^2}{2} + \cdots + \frac{A_c^N t^N}{N!}, \quad (27)$$

which leads to

$$A_D = I + A_c T_s + \frac{(A_c T_s)^2}{2!} + \frac{(A_c T_s)^3}{3!} + \frac{(A_c T_s)^4}{4!} + \cdots, \quad (28)$$

$$B_D = T_s + \frac{A_c T_s^2}{2!} + \frac{A_c^2 T_s^3}{3!} + \frac{A_c^3 T_s^4}{4!} + \cdots, \quad (29)$$

$$Q_D = B_c Q_c B_c^T T_s + \frac{(A_c B_c Q_c B_c^T + B_c Q_c B_c^T A_c^T) T_s^2}{2!} + \cdots. \quad (30)$$

It is common practice ([4] – [6]) to truncate the above series after terms linear in  $T_s$ . Some higher order terms can be retained in applications where parameter accuracy is critical. Since the limit as  $N \rightarrow \infty$  of  $T_s^N / N!$  is 0, the above series are valid for any value of  $T_s$ . However, the sample period needs to be sufficiently small, otherwise the above discretisations will be erroneous. According to the Nyquist-Shannon sampling theorem, the sampling rate is required to be at least

---

“We are more easily persuaded, in general, by the reasons we ourselves discover than those which are given to us by others.” *Blaise Pascal*

twice the highest frequency component of the continuous-time signal. In respect of (17), the output map may be written as

$$y(kT_s) = C_c x(kT_s) + D_c w(kT_s) \quad (31)$$

and thus

$$C_D = C_c, \quad (32)$$

$$D_D = D_c. \quad (33)$$

Following the approach of [7], it is assumed that the continuous-time signals are integrated between samples, for example, the discretised measurement noise is

$$v(kT_s) = \frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} v(\tau) d\tau. \text{ Then the corresponding measurement noise covariance}$$

is

$$R_D = \frac{1}{T_s^2} \int_{kT_s}^{(k+1)T_s} R_c d\tau = \frac{1}{T_s} R_c. \quad (34)$$

In some applications, such as inertial and satellite navigation [8], the underlying dynamic equations are in continuous-time, whereas the filters are implemented in discrete-time. In this case, any underlying continuous-time equations together with (28) – (30) can be calculated within a high rate foreground task, so that the discretised state-space parameters will be sufficiently accurate. The discrete-time filter recursions can then be executed within a lower rate background task.

### 2.2.11 Asymptotic Stability

Consider a discrete-time, linear, time-invariant system  $\mathcal{G}$  that operates on an input process  $w$  and produces an output process  $y$ . The system  $\mathcal{G}$  is said to be asymptotically stable if the output remains bounded, that is,  $y \in \ell_2$ , for any input  $w \in \ell_2$ . Two equivalent conditions for  $\mathcal{G}$  to be asymptotically stable are as follows.

- (i) The  $i$  eigenvalues of the system's state matrix are inside the unit circle, that is, for  $A_i$  of (11),  $|\lambda_i(A)| < 1$ .
- (ii) The  $i$  poles of the system's transfer function are inside the unit circle, that is, for  $\alpha_i$  of (9),  $|\alpha_i| < 1$ .

**Example 7.** A state-space system having  $A = -0.5$ ,  $B = C = 1$  and  $D = 0$  is stable, since  $\lambda(A) = 0.5$  is in the unit circle. Equivalently, the corresponding transfer function  $G(z) = (z + 0.5)^{-1}$  has a pole at  $z = -0.5$  which is inside the unit circle and so the system is stable.

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“Eighty percent of success is showing up.” (Woody) Allen Stewart Konigsberg

### 2.2.12 Adjoint Systems

Consider again a linear system  $\mathcal{G}$  be a linear system operating on an input  $w \in \mathbb{R}^{m \times N}$  and produces an output  $\mathcal{G} w \in \mathbb{R}^{p \times T}$ . Then  $\mathcal{G}^H$ , the adjoint of  $\mathcal{G}$ , is the unique linear system which produces an output signal  $\mathcal{G}^H \alpha \in \mathbb{R}^{m \times T}$ , such that  $\langle \alpha, \mathcal{G} w \rangle = \langle \mathcal{G}^H \alpha, w \rangle$ , for all  $\alpha \in \mathbb{R}^{p \times N}$  and  $w \in \mathbb{R}^{m \times N}$ . The following derivation is a simplification of the time-varying version that appears in [9].

**Lemma 1 (State-space representation of an adjoint system):** Suppose that a discrete-time linear time-invariant system  $\mathcal{G}$  is described by

$$x_{k+1} = Ax_k + Bw_k, \quad (35)$$

$$y_k = Cx_k + Dw_k, \quad (36)$$

with  $x_0 = 0$ . The adjoint  $\mathcal{G}^H$  is the linear system having the realisation

$$\zeta_{k-1} = A^T \zeta_k - C^T \alpha_k, \quad (37)$$

$$\beta_k = -B^T \zeta_k + D^T \alpha_k, \quad (38)$$

with  $\zeta_T = 0$ .

**Proof:** The system (35) – (36) can be written equivalently

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 0(t) \\ y(t) \end{bmatrix} \quad (39)$$

with  $x_0 = 0$ . Thus

$$\begin{aligned} \langle y, \mathcal{G} w \rangle &= \left\langle \begin{bmatrix} \zeta \\ \alpha \end{bmatrix}, \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle \\ &= \sum_{k=1}^N \zeta_k^T x_{k+1} - \sum_{k=1}^N \zeta_k^T (Ax_k + Bw_k) + \sum_{k=1}^N \alpha_k^T (Cx_k + Dw_k) \end{aligned} \quad (40)$$

$$\begin{aligned} &= \left\langle \begin{bmatrix} z^{-1}I - A^T & C^T \\ -B^T & D^T \end{bmatrix} \begin{bmatrix} \zeta \\ \alpha \end{bmatrix}, \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle \\ &= \langle \mathcal{G}^H \alpha, w \rangle \end{aligned} \quad (41)$$

where  $\mathcal{G}^H$  is given by (37) – (38).

---

“There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact.” *Samuel Langhorne Clemens aka. Mark Twain*

Thus, the adjoint of a discrete-time system having the parameters  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is a system with parameters  $\begin{bmatrix} A^T & -C^T \\ -B^T & D^T \end{bmatrix}$ . Adjoint systems have the property  $(\mathcal{G}^H)^H = \mathcal{G}$ . The adjoint of the transfer function matrix  $G(z)$  is denoted as  $G^H(z)$  and is defined by the transfer function matrix

$$G^H(z) = G^T(z^{-1}). \quad (42)$$

**Example 8.** Suppose that a system  $\mathcal{G}$  has the state-space parameters  $A = -0.5$  and  $B = C = D = 1$ . From Lemma 1, an adjoint system has the state-space parameters  $A = -0.5$ ,  $B = C = -1$ ,  $D = 1$  and the corresponding transfer function is  $G^H(z) = 1 + (z^{-1} + 0.5)^{-1} = (3z + 2)(z + 2)^{-1}$ , which is unstable and non-minimum-phase. Alternatively, the adjoint of  $G(z) = 1 + (z + 0.5)^{-1} = (z + 1.5)(z + 0.5)^{-1}$  can be obtained using (42), namely,  $G^H(z) = G^T(z^{-1}) = (3z + 2)(z + 2)^{-1}$ .

### 2.2.13 Causal Systems

A causal system is a system whose output depends exclusively on past and current inputs and outputs.

**Example 9.** Consider  $x_{k+1} = 0.3x_k + 0.4x_{k-1} + w_k$ . Since the output  $x_{k+1}$  depends only on past states  $x_k, x_{k-1}$ , and past inputs  $w_k$ , this system is causal.

**Example 10.** Consider  $x_k = 0.3x_{k+1} + 0.4x_k + w_{k+1}$ . Since the output  $x_k$  depends on future outputs  $x_{k+1}$  and future  $w_{k+1}$  inputs, this system is non-causal.

### 2.2.14 Realising Unstable System Components

Unstable system components are termed unrealisable because their outputs are not in  $\ell_2$ , that is, they are unbounded. In other words, unstable systems cannot produce a useful output. However, an unstable causal component can be realised as a stable non-causal or backwards component. Consider the system  $\mathcal{G}$  (35) – (36) in which the eigenvalues of  $A$  all lie outside the unit circle. In this case, a stable adjoint system  $\beta = \mathcal{G}^H \alpha$  can be realised by the following three-step procedure.

- (i) Time-reverse the input signal  $\alpha_k$ , that is, construct  $\alpha_\tau$ , where  $\tau = N - k$  is a time-to-go variable.
- (ii) Realise the stable system  $\mathcal{G}^T$

$$\zeta_{\tau+1} = A^T \zeta_{\tau} + C^T \alpha_{\tau}, \quad (43)$$

$$\beta_{\tau} = B^T \zeta_{\tau} + D^T \alpha_{\tau}, \quad (44)$$

with  $\zeta_T = 0$ .

(iii) Time-reverse the output signal  $\beta_{\tau}$ , that is, construct  $\beta_k$ .

Thus if a system consists of a cascade of stable and unstable components, it can be realised by a combination of causal and non-causal components. This approach will be exploited in the realisation of smoothers subsequently.

**Example 1.** Suppose that it is desired to realise the system  $G(z) = G_2^H(z)G_1(z)$ , in which  $G_1(z) = (z + 0.6)^{-1}$ ,  $G_2^H(z) = z(0.9z + 1)^{-1}$ , that is,  $G_2(z) = (z + 0.9)^{-1}$ . This system can be realised using the processes shown in Fig. 2.

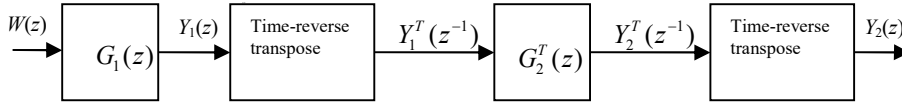


Fig. 2. Realising an unstable  $G(z) = G_2^H(z)G_1(z)$ .

### 2.2.15 Power Spectral Density

Consider again a linear, time-invariant system  $y = \mathcal{G} w$  and its corresponding transfer function matrix  $G(z)$ . Then  $\Phi_{yy}(z)$ , the power spectral density of  $y$ , is given by

$$\Phi_{yy}(z) = G Q G^H(z), \quad (45)$$

which has the property  $\Phi_{yy}(z) = \Phi_{yy}(z^{-1})$ . From Parseval's Theorem (5), the average total energy of  $y(t)$  is given by

$$\int_{-e^{-j\omega T}}^{e^{j\omega T}} \Phi_{yy}(z) dz = \int_{-\infty}^{\infty} |y_k|^2 dk = \|y(t)\|_2^2 = E\{y^T(t)y(t)\}, \quad (46)$$

which equals the area under the power spectral density curve.

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“Knowledge rests not upon truth alone, but on error also.” *Carl Gustav Jung*

## 2.2.16 Spectral Factorisation

To avoid confusion with the  $z$ -transform variable, denote the noisy measurements of  $y(z) = G(z)w(z)$  by

$$u(z) = y(z) + v(z), \quad (47)$$

where  $v(z) \in \mathbb{R}^p$  is the  $z$ -transform of an independent, zero-mean, stationary, white measurement noise process with  $E\{v_j v_k^T\} = R\delta_{jk}$ . Let

$$\Phi_{uu}(z) = GQG^H(z) + R \quad (48)$$

denote the spectral density matrix of the measurements  $u(t)$ . A discrete-time transfer function is said to be minimum phase if its zeros lies inside the unit circle. Conversely, transfer functions having outside-unit-circle-zeros are known as non-minimum phase.

Suppose that  $\Phi_{uu}(z)$  is a spectral density matrix of transfer functions possessing equal order numerator and denominator polynomials that do not have roots on the unit circle. Then the spectral factor matrix  $\Delta(z)$  satisfies the following.

- (i)  $\Delta(z) \Delta^H(z) = \Phi_{uu}(z)$ .
- (ii)  $\Delta(z)$  is causal, that is, the poles of  $\Delta(z)$  are inside the unit circle.
- (iii)  $\Delta^{-1}(z)$  is causal, that is, the zeros of  $\Delta(z)$  which are the poles of  $\Delta^{-1}(z)$  are inside the unit circle.

The problem of spectral factorisation within discrete-time Wiener filtering problems is studied in [10]. The roots of the transfer function polynomials need to be sorted into those inside the unit circle and those outside the unit circle. Spectral factors can be found using Levinson-Durbin and Schur algorithms, Cholesky decomposition, Riccati equation solution [11] and Newton-Raphson iteration [12].

**Example 2.** Applying the Bilinear Transform (15) to the continuous-time low-pass plant  $G(s) = (s + 1)^{-1}$  for a sample frequency of 2 Hz yields  $G(z) = 0.2(z+1)(z-0.6)^{-1}$ . With  $Q = R = 1$ , the measurement spectral density (48) is  $\Phi_{uu}(z) = \frac{(1.08z - 0.517)}{(z - 0.6)} \times \frac{(-0.517z + 1.08)}{(-0.6z - 1.0)}$ . By inspection,  $\Delta(z) = (1.08z - 0.517)(z - 0.6)^{-1}$  has inside-unit-circle-poles and zeros that satisfy  $\Delta(z)\Delta^H(z) = \Phi_{uu}(z)$ .

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“It is still an unending source of surprise for me how a few scribbles on a blackboard or on a piece of paper can change the course of human affairs.” *Stanislaw Ulam*



**Example 3.** Consider the high-pass plant  $G(z) = 4.98(z - 0.6)(z + 0.99)^{-1}$  and  $Q = R = 1$ . The spectral density is  $\Phi_{uu}(z) = \frac{(5.39z - 2.58)}{(z + 0.99)} \times \frac{(-2.58z + 5.39)}{(0.99z - 1.0)}$ . Thus the stable, minimum phase spectral factor is  $\Delta(z) = (5.39z - 2.58)(z + 0.99)^{-1}$ , since it has inside-unit-circle-poles and zeros.

### 2.2.17 Calculating Causal Parts

Suppose that a discrete-time transfer function has the form

$$\begin{aligned} G(z) &= c_0 + \sum_{i=1, |a_i| < 1}^n \frac{d_i}{z - a_i} + \sum_{j=1, |b_j| > 1}^m \frac{e_j}{z - b_j} \\ &= c_0 + G_{iucp}(z) + G_{oucp}(z), \end{aligned} \quad (49)$$

where  $c_0, d_i, e_j \in \mathbb{R}$ ,  $G_{iucp}(z) = \sum_{i=1, |a_i| < 1}^n \frac{d_i}{z - a_i}$  is the sum of partial fractions having

inside-unit-circle-poles and  $G_{oucp}(z) = \sum_{j=1, |b_j| > 1}^m \frac{e_j}{z - b_j}$  is the sum of partial fractions

having outside-unit-circle-poles. Assume that the roots of  $G(z)$  are distinct and do not lie on the unit circle. In this case the partial fraction coefficients  $d_i$  and  $e_j$  within (49) can be calculated from the numerator and denominator polynomials of  $G(z)$  via  $d_i = (z - a_i)G(z)|_{z=a_i}$  and  $e_j = (z - b_j)G(z)|_{z=b_j}$ . Previously, in

continuous-time, the convention was to define constants to be causal. This is consistent with ensuring that the non-causal part of the discrete-time transfer function is zero at  $z = 0$ . Thus, the non-causal part of  $G(z)$ , denoted by  $\{G(z)\}_-$ , is obtained as

$$\{G(z)\}_- = G_{oucp}(z) - G_{oucp}(0) \quad (50)$$

and the causal part of  $G(z)$ , denoted by  $\{G(z)\}_+$ , is whatever remains, that is,

$$\begin{aligned} \{G(z)\}_+ &= G(z) - \{G(z)\}_- \\ &= c_0 + G_{iucp}(z) + G_{oucp}(0). \end{aligned} \quad (51)$$

Hence, the causal part of transfer function can be found by carrying out the following three steps.

- (i) If the transfer function is not strictly proper, that is, if the order of the numerator not less than the degree of the denominator, perform synthetic division to extract the constant term.

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“I shall try to correct errors when shown to be errors; and I shall adopt new views so fast as they shall appear to be true views.” *Abraham Lincoln*

- (ii) Expand out the (strictly proper) transfer function into the sum of partial fractions (49).
- (iii) Obtain the causal part from (51), namely, take the sum of the constant term, the partial fractions with inside-unit-circle-poles and the partial fractions with outside-unit-circle-poles evaluated at  $z = 0$ .

**Example 4.** Consider the strictly proper transfer function  $G(z) = \frac{3z+3.2}{z^2+2.6z+1.2} = \frac{3z+3.2}{(z+0.6)(z+2)} = \frac{1}{z+0.6} + \frac{2}{z+2}$ . It follows from (50) and (51) that  $\{G(z)\}_- = \frac{2}{z+2} - 1 = \frac{-z}{z+2}$  and  $\{G(z)\}_+ = \frac{1}{z+0.6} + 1 = \frac{z+1.6}{z+0.6}$ , respectively. It is easily verified that  $G(z) = \{G(z)\}_+ + \{G(z)\}_-$ .

**Example 5.** Consider the proper transfer function  $G(z) = \frac{2z^2+8.2z+5.6}{z^2+2.6z+1.2}$ . Carrying out synthetic division results in  $G(z) = 2 + \frac{1}{z+0.6} + \frac{2}{z+2}$ . It follows from (50) and (51) that  $\{G(z)\}_- = \frac{2}{z+2} - 1 = \frac{-z}{z+2}$  and  $\{G(z)\}_+ = \frac{1}{z+0.6} + 1 + 2 = \frac{3z+2.8}{z+0.6}$ , respectively.

## 2.3 Minimum-Mean-Square-Error Filtering

### 2.3.1 Filter Derivation

This section derives the optimal non-causal minimum-mean-square-error solution for the problem configuration of Fig. 3. The derivation is identical to the continuous-time case which is presented in Chapter 1. It is assumed that the parameters of the transfer function  $G_2(z) = C_2(zI - A)^{-1}B + D_2$  are known. Let  $Y_2(z)$ ,  $W(z)$ ,  $V(z)$  and  $U(z)$  denote the z-transform of a system's output, process noise, measurement noise and observations, respectively. Then it follows from (47) that the z-transform of the measurements is

$$U(z) = Y_2(z) + V(z). \quad (52)$$

Consider a fictitious reference system  $G_1(z) = C_1(zI - A)^{-1}B + D_1$  as shown in Fig. 3. The problem is to design a stable filter transfer function  $H(z)$  to calculate

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"I think anybody who doesn't think I'm smart enough to handle the job is underestimating." *George Walker Bush*

estimates  $\hat{Y}_1(z) = H(z)U(z)$  of  $Y_1(z)$  so that the energy  $\int_{-j\infty}^{j\infty} \tilde{Y}(z)\tilde{Y}^H(z)dz$  of the estimation error

$$\tilde{Y}(z) = \hat{Y}_1(z) - Y_1(z) \quad (53)$$

is minimised. It can be seen from Fig. 3 that the estimation error is generated by the system

$$\tilde{Y}(z) = -[H(z) \quad H_2G(z) - G_1(z)] \begin{bmatrix} V(z) \\ W(z) \end{bmatrix}. \quad (54)$$

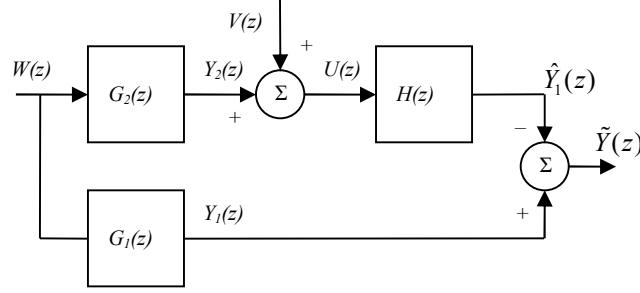


Fig. 3. The general z-domain filtering problem.

The error power spectrum density matrix is given by the covariance of  $\tilde{Y}(z)$ , that is,

$$\begin{aligned} \Phi_{\tilde{Y}\tilde{Y}}(z) &= \tilde{Y}(z)\tilde{Y}^H(z) \\ &= [H(z) \quad HG_2(z) - G_1(z)] \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} H^H(z) \\ G_2^H H^H(z) - G_1^H(z) \end{bmatrix} \\ &= G_1 Q G_1^H(z) - G_1 Q G_2^H H^H(z) - H G_2 Q G_1^H(z) + H \Delta \Delta^H H^H(z), \end{aligned} \quad (55)$$

where

$$\Delta \Delta^H(z) = G_2 Q G_2^H(z) + R \quad (56)$$

is the spectral density matrix of the measurements. Completing the square within (55) yields

$$\begin{aligned} \Phi_{\tilde{Y}\tilde{Y}}(z) &= G_1 Q G_1^H(z) - G_1 Q G_2^H (\Delta \Delta^H)^{-1} G_2 Q G_1^H(z) \\ &\quad + (H \Delta(z) - G_1 Q G_2^H \Delta^{-H}(z))(H \Delta(z) - G_1 Q G_2^H \Delta^{-H}(z))^H, \end{aligned} \quad (57)$$

---

“We are all born ignorant, but must work hard to remain stupid.” *Benjamin Franklin*

in which  $\Delta^{-H}(z) = (\Delta^H)^{-1}(z)$ . It follows that the total energy of the error signal can be expressed as

$$\begin{aligned} & \int_{-e}^e \int_{-e}^e \Phi_{\tilde{Y}\tilde{Y}}(z) dz \\ &= \int_{-e}^e \int_{-e}^e G_1 Q G_1^H(z) - G_1 Q G_2^H (\Delta \Delta^H)^{-1} G_2 Q G_1^H(z) dz \\ &+ \int_{-e}^e \int_{-e}^e (H \Delta(z) - G_1 Q G_2^H \Delta^{-H}(z)) (H \Delta(z) - G_1 Q G_2^H \Delta^{-H}(z))^H dz. \end{aligned} \quad (58)$$

The first term on the right-hand-side of (58) is independent of  $H(z)$  and represents a lower bound of  $\int_{-e}^e \int_{-e}^e \Phi_{\tilde{Y}\tilde{Y}}(z) dz$ . The second term on the right-hand-side of (109) may be minimised by a judicious choice for  $H(z)$ .

**Theorem 1:** *The optimal solution for the above linear time-invariant estimation problem with measurements (103) and error (104) is*

$$H(z) = G_1 Q G_2^H \Delta^{-H} \Delta^{-1}(z), \quad (59)$$

which minimises  $\int_{-e}^e \int_{-e}^e \Phi_{\tilde{Y}\tilde{Y}}(z) dz$ .

**Proof:** *The result follows by setting  $H \Delta(z) - G_1 Q G_2^H \Delta^{-H}(z)$  equal to the zero matrix within (58).*

By Parseval's theorem, the minimum mean-square-error solution (59) also minimises . The solution (59) is non-causal because the factor possesses outside-unit-circle poles. This optimal non-causal solution is actually a smoother, which can be realised by a combination of forward and backward processes.

The transfer function matrix of the optimal causal solution or filter is obtained by setting the setting the causal part of equal to the zero matrix, resulting in = , that is = , which implies

$$H(z) = \left\{ G_1 Q G_2^H (\Delta^H)^{-1} \right\}_+ \Delta^{-1}(z). \quad (60)$$

### 2.3.2 Output Estimation

In output estimation, it is desired to estimate the output  $Y_2(z)$  from the measurements  $U(z)$ , in which case the reference system is the same as the

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“He who knows nothing is closer to the truth than he whose mind is filled with falsehoods and errors.”  
Thomas Jefferson

generating system, as shown in Fig. 4. The optimal non-causal solution (59) with  $G_1(z) = G_2(z)$  becomes

$$H_{OE}(z) = G_2 Q G_2^H \Delta^{-H} \Delta^{-1}(z). \quad (61)$$

Substituting  $G_2 Q G_2^H(z) = \Delta \Delta^H(z) - R$  into (61) leads to the alternative form

$$\begin{aligned} H_{OE}(z) &= (\Delta \Delta^H - R)(\Delta \Delta^H)^{-1}(z) \\ &= I - R \Delta^{-H} \Delta^{-1}(z). \end{aligned} \quad (62)$$

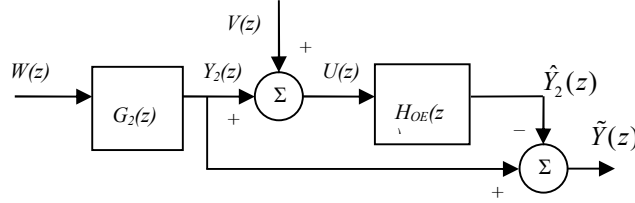


Fig. 4. The z-domain output estimation problem.

The solutions (61) and (62) are non-causal since  $G^H(z)$  and  $\Delta^{-H}(z)$  are non-causal. The optimal smoother or non-causal filter for output estimation is obtained by substituting  $G_1(z) = G_2(z)$  into (60), namely,

$$H_{OE}(z) = \{G_2 Q G_2^H \Delta^{-H}\}_+ \Delta^{-1}(z). \quad (63)$$

An alternative form arises by substituting  $G Q G^H(z) = \Delta \Delta^H(z) - R$  into (63), which results in

$$\begin{aligned} H_{OE}(z) &= \{\Delta(z) - R \Delta^{-H}\}_+ \Delta^{-1}(z) \\ &= I - R \{\Delta^{-H}\}_+ \Delta^{-1}(z). \end{aligned} \quad (64)$$

In [10], it is recognised that  $\{\Delta^H(z)\}_+ = \lim_{z \rightarrow \infty} \Delta(z)$ , which is equivalent to  $\{\Delta^H(z)\}_+ = \Delta^H(0)$ . It follows that

$$H_{OE}(z) = I - R \Delta^{-H}(0) \Delta^{-1}(z), \quad (65)$$

which eliminates the need for calculating causal parts.

**Example 6.** Consider  $G_2(z) = (z + 0.2)(z + 0.5)^{-1}$  together with  $R = Q = 1$ . The spectral factor is  $\Delta(z) = (1.43z + 0.489)(z + 0.5)^{-1}$ , which leads to  $G_2 Q G_2^H \Delta^{-H}(z) = (0.2z^2 + 1.04z + 0.2)(0.489z^2 + 1.67z + 0.716)^{-1}$  and  $\{G_2 Q G_2^H \Delta^{-H}(z)\}_+ = (0.734z$

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“There is much pleasure to be gained from useless knowledge.” *Bertrand Arthur William Russell*

+ 0.14)(z + 0.5)<sup>-1</sup>. Hence, from (63),  $H_{OE}(z) = (0.513z + 0.098)(z + 0.341)^{-1}$ . The same solution can be calculated using  $\Delta^{-H}(0) = 0.698$  within (65).

When the measurement noise becomes negligibly small, the output estimator approaches a short circuit, that is,

$$\lim_{R \rightarrow 0, e^{j\omega T} \rightarrow 0} |H_{OE}(z)| = I, \quad (66)$$

The above observation can be verified by substituting  $R = 0$  into (65). This asymptote is consistent with intuition, that is, when the measurements are perfect, output estimation will be superfluous.

**Example 7.** Substituting  $R = 0.001$  within Example 15 yields the filter  $H(z) = (0.999z + 0.2)(z + 0.2)^{-1}$ , which illustrates the low measurement noise asymptote (66).

### 2.3.3 Input Estimation

In input estimation or equalisation problems,  $G_2(z)$  is known as the channel model and it is desired to estimate the input process  $w(t)$ , as depicted in Fig. 5. The simplification of the optimum non-causal solution (59) for the case of  $G_1(z) = I$  is

$$H_{IE}(z) = QG_2^H \Delta^{-H} \Delta^{-1}(z), \quad (67)$$

Assume that: the channel model  $G_2(z)$  is proper, that is, the order of the numerator is the same as the order of the denominator; and that the channel model  $G_2(z)$  is stable and minimum phase, that is, its poles and zeros are inside the unit circle. The causal equaliser for proper, stable, minimum-phase channels is obtained by substituting  $G_1(z) = I$  into (60)

$$\begin{aligned} H_{IE}(z) &= \{QG_2^H \Delta^{-H}\}_+ \Delta^{-1}(z) \\ &= QG_2^H(0) \Delta^{-H}(0) \Delta^{-1}(z). \end{aligned} \quad (68)$$

Under the above assumptions, the causal equaliser may be written equivalently as

$$\begin{aligned} H_{IE}(z) &= \{G_2^{-1} G_2 Q G_2^H \Delta^{-H}\}_+ \Delta^{-1}(z) \\ &= \{G_2^{-1} (\Delta \Delta^H - R) \Delta^{-H}\}_+ \Delta^{-1}(z) \end{aligned} \quad (69)$$

$$\begin{aligned} &= G_2^{-1} (I - R \{\Delta^{-H}\}_+ \Delta^{-1}(z)) \\ &= G_2^{-1} (I - R \Delta^{-H}(0) \Delta^{-1}(z)) \end{aligned} \quad (70)$$

Thus, the equaliser is equivalent to a product of the channel inverse and the output estimator. It follows that when the measurement noise becomes negligibly small, the equaliser estimates the inverse of the system model, that is,

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“Time is a great teacher, but unfortunately it kills all its pupils.” *Louis Hector Berlioz*

$$\lim_{R \rightarrow 0} H_{IE}(z) = G_2^{-1}(z), \quad (71)$$

The above observation follows by substituting  $R = 0$  into (69). In other words, if the channel model is invertible and signal to noise ratio is sufficiently high, the equaliser will estimate  $w(t)$ . When measurement noise is present then the solution trades off channel inversion and filtering. In the high measurement noise case, the equaliser approaches an open circuit, that is,

$$\lim_{Q \rightarrow 0, e^{j\omega T} \rightarrow 0} |H_{IE}(z)| = 0. \quad (72)$$

The above observation can be verified by substituting  $\Delta\Delta^H = R$  into (70). Thus, when the equalisation problem is dominated by measurement noise, the estimation error is minimised by ignoring the data.

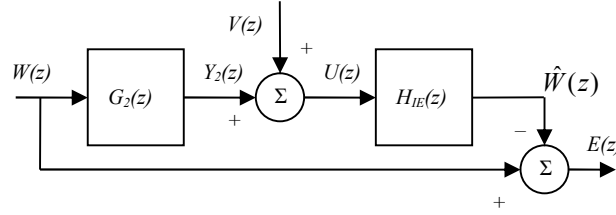


Fig. 5. The z-domain input estimation problem.

**Example 8.** Consider the high-pass plant  $G_2(s) = 100(s + 0.1)(s + 10)^{-1}$ . Application of the bilinear transform for a sample frequency of 2 Hz yields  $G_2(z) = (29.2857z - 27.8571)(z + 0.4286)^{-1}$ . With  $Q = 1$  and  $R = 0.001$ , the spectral factor is  $\Delta(z) = (29.2861z - 27.8568)(z + 0.4286)^{-1}$ . From (67),  $H_{IE}(z) = (z + 0.4286)(29.2861z - 27.8568)^{-1}$ , which is high-pass and illustrates (71).

**Example 9.** Applying the bilinear transform for a sample frequency of 2 Hz to the low-pass plant  $G_2(z) = (s + 10)(s + 0.1)^{-1}$  results in  $G_2(z) = (3.4146z - 1.4634)(z - 0.9512)^{-1}$ . With  $Q = 1$  and  $R = 0.001$ , the spectral factor is  $\Delta(z) = (3.4151z + 1.4629)(z - 0.9512)^{-1}$ . From (67),  $H_{IE}(z) = (z - 0.9512)(3.4156z + 1.4631)^{-1}$ , which is low pass and is consistent with (71).

## 2.4 Chapter Summary

Systems are written in the time-domain as difference equations

$$a_n y_{k-n} + a_{n-1} y_{k-n+1} + \dots + a_1 y_{k-1} + a_0 y_k = b_m w_{k-m} + b_{m-1} w_{k-m+1} + \dots + b_1 w_{k-1} + b_0 w_k,$$

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“If I have made any valuable discoveries, it has been owing more to patient attention, than to any other talent.” *Sir Isaac Newton*

which can be expressed as polynomial transfer functions in the z-transform variable

$$Y(z) = \left[ \frac{b_m z^{-m} + b_{m-1} z^{-m+1} + \dots + b_1 z^{-1} + b_0}{a_n z^{-n} + a_{n-1} z^{-n+1} + \dots + a_1 z^{-1} + a_0} \right] W(z) = G(z)W(z).$$

It can be seen that knowledge of a system's differential equation is sufficient to identify its transfer function. The optimal Wiener solution minimises the energy of the error (and the mean-square-error), and the main results are summarised in Table 1. The noncausal (or smoother) solution has unstable factors and can only be realised by a combination of forward and backward processes.

	ASSUMPTIONS	MAIN RESULTS
Signals and systems	$E\{w_k\} = E\{W(z)\} = E\{v_k\} = E\{V(z)\} = 0$ . $E\{w_k w_k^T\} = E\{W(z)W^T(z)\} = Q > 0$ and $E\{v_k v_k^T\} = E\{V(z)V^T(z)\} = R > 0$ are known. $A$ , $B$ , $C_1$ , $C_2$ , $D_1$ and $D_2$ are known. $G_1(z)$ and $G_2(z)$ are stable, i.e., $ \lambda_i(A)  < 1$ .	$G_1(z) = C_1(zI - A)^{-1}B + D_1$ $G_2(z) = C_2(zI - A)^{-1}B + D_2$
Spectral factorisation	$\Delta(z)$ and $\Delta^{-1}(z)$ are stable, i.e., the poles and zeros of $\Delta(z)$ are inside the unit circle.	$\Delta\Delta^H(z) = G_2 Q G_2^H(z) + R$
Non-causal solution		$H(z) = G_1 Q G_2^H (\Delta^H)^{-1} \Delta^{-1}(z)$
Causal solution		$H(z) = \{G_1 Q G_2^H (\Delta^H)^{-1}\}_+ \Delta^{-1}(z)$

Table 1. Main results for the discrete-time general filtering problem.

It is noted that  $\{\Delta^H(z)\}_+ = \lim_{z \rightarrow \infty} \Delta(z) = \Delta^H(0)$ , which can simplify calculating causal parts. For example, in output estimation problems where  $G_1(z) = G_2(z)$ , the minimum-mean-square-error solution is  $H_{OE}(z) = I - R\Delta^{-H}(0)\Delta^{-1}(z)$ . In the single-input-single-output case, when the measurement noise becomes negligible, the output estimator approaches a short circuit. Conversely, when the single-input-

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“All our knowledge has its origin in our perceptions.” *Leonardo da Vinci*



single-output problem is dominated by measurement noise, the output estimator approaches an open circuit.

In input estimation problems,  $G_1(z) = I$ . If the channel model is invertible, the optimal causal equaliser is given by  $H_{IE}(z) = QG_2^H(0)\Delta^{-H}(0)\Delta^{-1}(z)$ . When the measurement noise becomes negligible, that is,  $\Delta^{-1}(z) \approx G_2^H(z)$ , the optimal equaliser approaches the channel inverse. Conversely, when the problem is dominated by measurement noise, the equaliser approaches an open circuit.

## 2.5 Problems

**Problem 1.** Consider the error spectral density matrix

$$\Phi_{ee}(z) = [H\Delta - G_1QG_2^H(\Delta^H)^{-1}][H\Delta - G_1QG_2^H(\Delta^H)^{-1}]^H(z) \\ + [G_1QG_1^H - G_1QG_2^H(\Delta\Delta^H)^{-1}G_2QG_1^H](z).$$

- Derive the optimal non-causal solution.
- Derive the optimal causal filter from (a).
- Derive the optimal non-causal output estimator.
- Derive the optimal causal filter from (c).
- Derive the optimal non-causal input estimator.
- Derive the optimal causal equaliser assuming that the channel inverse exists.

**Problem 2.** Derive the asymptotes for the following single-input-single-output estimation problems.

- Non-causal output estimation at  $R = 0$ .
- Non-causal output estimation at  $Q = 0$ .
- Causal output estimation at  $R = 0$ .
- Causal output estimation at  $Q = 0$ .
- Non-causal input estimation at  $R = 0$ .
- Non-causal input estimation at  $Q = 0$ .
- Causal input estimation at  $R = 0$ .
- Causal input estimation at  $Q = 0$ .

**Problem 3.** In respect of the output estimation problem with  $G(z) = (z - \beta)(z - \alpha)^{-1}$ ,  $\alpha = -0.3$ ,  $\beta = -0.5$  and  $Q=1$ , verify the following.

- $R = 10$  yields  $H(z) = (0.0948z + 0.0272)(z + 0.4798)^{-1}$ .
- $R = 1$  yields  $H(z) = (0.5059z + 0.1482)(z + 0.3953)^{-1}$ .
- $R = 0.1$  yields  $H(z) = (0.90941z + 0.2717)(z + 0.3170)^{-1}$ .
- $R = 0.01$  yields  $H(z) = (0.9901z + 0.2969)(z + 0.3018)^{-1}$ .
- $R = 0.001$  yields  $H(z) = (0.9990z + 0.2997)(z + 0.3002)^{-1}$ .

**Problem 4.** In respect of the input estimation problem with  $G(z) = (z - \beta)(z - \alpha)^{-1}$ ,  $\alpha = -0.9$ ,  $\beta = -0.1$  and  $Q=1$ , verify the following.

- $R = 10$  yields  $H(z) = (z + 0.1)(11.5988z + 1.9000)^{-1}$ .

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“Statistics is the main of all inaccurate studies” *Edmond de Goncourt*

- (b)  $R = 1$  yields  $H(z) = (z + 0.1)(2.4040z + 1.0000)^{-1}$ .  
 (b)  $R = 0.1$  yields  $H(z) = (z + 0.1)(1.2468z + 0.9100)^{-1}$ .  
 (d)  $R = 0.01$  yields  $H(z) = (z + 0.1)(1.0381z + 0.9010)^{-1}$ .  
 (e)  $R = 0.001$  yields  $H(z) = (z + 0.1)(1.043z + 0.9001)^{-1}$ .

## 2.6 Glossary

The following terms have been introduced within this section.

$k$	The integer-valued time variable. For example, $k \in (-\infty, \infty)$ and $k \in (0, \infty)$ denote $-\infty < k < \infty$ and $0 \leq k < \infty$ , respectively.
$w_k \in \mathbb{R}^n$	A discrete-time, real-valued, $n$ -element stochastic input signal.
$w$	The set of $w_k$ over a prescribed interval.
$y = \mathcal{G}w$	The output of a linear system $\mathcal{G}$ that operates on an input signal $w$ .
$A, B, C, D$	Time-invariant state space matrices of appropriate dimension. The system $\mathcal{G}$ is assumed to have the realisation $x_{k+1} = Ax_k + Bw_k$ , $y_k = Cx_k + Dw_k$ in which $w_k$ is known as the process noise or input signal.
$v_k$	A stationary stochastic measurement noise signal.
$\delta_{jk}$	The Kronecker delta function.
$Q$ and $R$	Time-invariant covariance matrices of stochastic signals $w_k$ and $v_k$ , respectively.
$Y(z)$	The z-transform of a continuous-time signal $y_k$ .
$G(z)$	The transfer function matrix of the system $\mathcal{G}$ . For example, the transfer function matrix of the system $x_{k+1} = Ax_k + Bw_k$ , $y_k = Cx_k + Dw_k$ is given by $G(z) = C(zI - A)^{-1}B + D$ .
$\langle v, w \rangle$	The inner product of two discrete-time signals $v$ and $w$ which is defined by $\langle v, w \rangle = \sum_{k=-\infty}^{\infty} v_k^T w_k$ .
$\ w\ _2$	The 2-norm of the discrete-time signal $w$ which is defined by $\ w\ _2 = \sqrt{\langle w, w \rangle} = \sqrt{\sum_{k=-\infty}^{\infty} w_k^T w_k}$ .
$\ell_2$	The set of continuous-time signals having finite 2-norm, which is known as the Lebesgue 2-space (see [3]).
Asymptotic stability	A linear discrete-time system $\mathcal{G}$ is said to be asymptotically stable if its output $y \in \ell_2$ for any $w \in \ell_2$ . If the real parts of the state matrix eigenvalues are inside the unit circle or equivalently if the real part of transfer function's poles are inside the unit circle then the system is stable.
$T_s$	Sample period.

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“If your result needs a statistician then you should design a better experiment.” *Baron Ernest Rutherford*

$\mathcal{G}^H$	The adjoint of $\mathcal{G}$ . The adjoint of a system having the state-space parameters $\{A, B, C, D\}$ is a system parameterised by $\{A^T, -C^T, -B^T, D^T\}$ .
$G^H(z)$	The adjoint (or Hermitian transpose) of the transfer function matrix $G(z)$ .
$\Phi_{\tilde{Y}\tilde{Y}}(z)$	The spectral density matrix of the error $\tilde{Y}(z)$ .
$\Delta(z)$	The spectral factor of $\Phi_{uu}(z)$ which satisfies $\Delta\Delta^H(z) = GG^H(z) + R$ . For brevity denote $\Delta^{-H}(z) = (\Delta^H)^{-1}(z)$ .
$G^{-1}(z)$	The inverse of the transfer function matrix $G(z)$ .
$G^{-H}(z)$	The inverse of the adjoint transfer function matrix $G^H(z)$ .
$\{G(z)\}_+$	The causal part of the transfer function matrix $G(z)$ .
$H(z)$	Transfer function matrix of the minimum mean-square-error solution.
$H_{OE}(z)$	Transfer function matrix of the minimum mean-square-error solution specialised for output estimation.
$H_{IE}(z)$	Transfer function matrix of the minimum mean-square-error solution specialised for input estimation.

## 2.2 References

- [1] N. Wiener, *Extrapolation, interpolation and smoothing of stationary time series with engineering applications*, The MIT Press, Cambridge Mass.; Wiley, New York; Chapman & Hall, London, 1949.
- [2] P. Masani, “Wiener’s Contributions to Generalized Harmonic Analysis, Prediction Theory and Filter Theory”, *Bulletin of the American Mathematical Society*, vol. 72, no. 1, pt. 2, pp. 73 – 125, 1966.
- [3] C. A. Desoer and M. Vidyasagar, *Feedback Systems : Input Output Properties*, Academic Press, N.Y., 1975.
- [4] F. L. Lewis, L. Xie and D. Pota, *Optimal and Robust Estimation With an Introduction to Stochastic Control Theory*, Second Edition, CRC Press, Taylor & Francis Group, 2008.
- [5] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York, 1971.
- [6] K. Ogata, *Discrete-time Control Systems*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1987.
- [7] S. Shats and U. Shaked, “Discrete-Time Filtering of Noise Correlated Continuous-Time Processes: Modelling and derivation of the Sampling Period Sensitivities”, *IEEE Transactions on Automatic Control*, vol. 36, no. 1, pp. 115 – 119, Jan. 1991.
- [8] P. G. Savage, *Strapdown Analytics*, Strapdown Associates, Maple Plain, Minnesota, USA, vol. 1 and 2, 2000.

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“There’s two possible outcomes: if the result confirms the hypothesis, then you’ve made a discovery. If the result is contrary to the hypothesis, then you’ve made a discovery.” *Enrico Fermi*

- [9] G. A. Einicke, “Optimal and Robust Noncausal Filter Formulations”, *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 1069 – 1077, Mar. 2006.
- [10] U. Shaked, “A transfer function approach to the linear discrete stationary filtering and the steady state discrete optimal control problems”, *International Journal of Control*, vol. 29, no. 2, pp 279 – 291, 1979.
- [11] A. H. Sayed and T. Kailath, “A Survey of Spectral Factorization Methods”, *Numerical Linear Algebra with Applications*, vol. 8, pp. 467 – 496, 2001.
- [12] H. J. Orchard and A. N. Wilson, “On the Computation of a Minimum-Phase Spectral Factor”, *IEEE Transactions on. Circuits and Systems I: Fundamental Theory and Applications*, vol. 50, no. 3, pp. 365 – 375, Mar. 2003.

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“One of the greatest discoveries a man makes, one of his greatest surprises, is to find he can do what he was afraid he couldn’t.” *Henry Ford*

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## 3. Continuous-Time, Minimum-Variance Filtering

### 3.1 Introduction

Rudolf E. Kalman studied discrete-time linear dynamic systems for his master's thesis at MIT in 1954. He commenced work at the Research Institute for Advanced Studies (RIAS) in Baltimore during 1957 and nominated Richard S. Bucy to join him in 1958 [1]. Bucy recognised that the nonlinear ordinary differential equation studied by an Italian mathematician, Count Jacopo F. Riccati, in around 1720, now called the Riccati equation, is equivalent to the Wiener-Hopf equation for the case of finite dimensional systems [1], [2]. In November 1958, Kalman recasted the frequency domain methods developed by Norbert Wiener and Andrei N. Kolmogorov in the 1940s to state-space form [2]. Kalman noted in his 1960 paper [3] that generalising the Wiener solution to nonstationary problems was difficult, which motivated his development of the optimal discrete-time filter in a state-space framework. He described the continuous-time version with Bucy in 1961 [4] and published a generalisation in 1963 [5]. Bucy later investigated the monotonicity and stability of the underlying Riccati equation [6]. The continuous-time minimum-variance filter is now commonly attributed to both Kalman and Bucy.

Compared to the Wiener Filter, Kalman's state-space approach has the following advantages.

- It is applicable to time-varying problems.
- As noted in [7], [8], the state-space parameters can be linearisations of nonlinear models.
- The burdens of spectral factorisation and pole-zero cancelation are replaced by the easier task of solving a Riccati equation.
- It is a more intuitive model-based approach in which the estimated states correspond to those within the signal generation process.

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“What a weak, credulous, incredulous, unbelieving, superstitious, bold, frightened, what a ridiculous world ours is, as far as concerns the mind of man. How full of inconsistencies, contradictions and absurdities it is. I declare that taking the average of many minds that have recently come before me ... I should prefer the obedience, affections and instinct of a dog before it.” *Michael Faraday*

Kalman's research at the RIAS was concerned with estimation and control for aerospace systems which was funded by the Air Force Office of Scientific Research. His explanation of why the dynamics-based Kalman filter is more important than the purely stochastic Wiener filter is that "Newton is more important than Gauss" [1]. The continuous-time Kalman filter produces state estimates  $\hat{x}(t)$  from the solution of a simple differential equation

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)(z(t) - C(t)\hat{x}(t)),$$

in which it is tacitly assumed that the model is correct, the noises are zero-mean, white and uncorrelated. It is straightforward to include nonzero means, coloured and correlated noises. In practice, the true model can be elusive but a simple (low-order) solution may return a cost benefit.

The Kalman filter can be derived in many different ways. In an early account [3], a quadratic cost function was minimised using orthogonal projections. Other derivation methods include deriving a maximum *a posteriori* estimate, using Itô's calculus, calculus-of-variations, dynamic programming, invariant imbedding and from the Wiener-Hopf equation [6] - [17]. This chapter provides a brief derivation of the optimal filter using a conditional mean (or equivalently, a least mean square error) approach.

The developments begin by introducing a time-varying state-space model. Next, the state transition matrix is defined, which is used to derive a Lyapunov differential equation. The Kalman filter follows immediately from a conditional mean formula. Its filter gain is obtained by solving a Riccati differential equation corresponding to the estimation error system. Generalisations for problems possessing deterministic inputs, correlated process and measurement noises, and direct feedthrough terms are described subsequently. Finally, it is shown that the Kalman filter reverts to the Wiener filter when the problems are time-invariant.

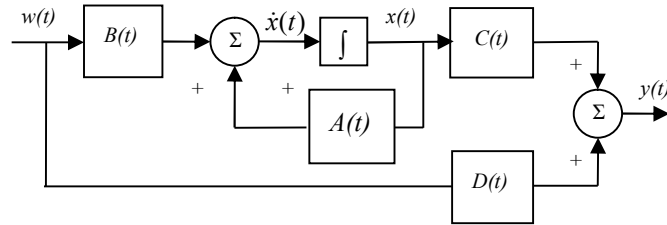


Fig. 1. The continuous-time system  $\mathcal{G}$  operates on the input signal  $w(t)$  and produces the output signal  $y(t)$ .

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"A great deal of my work is just playing with equations and seeing what they give." *Paul Arien Maurice Dirac*

## 3.1 Prerequisites

### 3.1.1 The Time-varying Signal Model

The focus initially is on time-varying problems over a finite time interval  $t \in [0, T]$ . A system  $\mathcal{G}$  is assumed to have the state-space representation

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (1)$$

$$y(t) = C(t)x(t) + D(t)w(t), \quad (2)$$

where  $A(t) \in \mathbb{R}^{n \times n}$ ,  $B(t) \in \mathbb{R}^{n \times m}$ ,  $C(t) \in \mathbb{R}^{p \times n}$ ,  $D(t) \in \mathbb{R}^{p \times m}$  and  $w(t) \in \mathbb{R}^m$  is a zero-mean white process noise with  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t - \tau)$ , in which  $\delta(t)$  is the Dirac delta function. This system is depicted in Fig. 1. In many problems of interest, signals are band-limited, that is, the direct feedthrough matrix,  $D(t)$ , is zero. Therefore, the simpler case of  $D(t) = 0$  is addressed first and the inclusion of a nonzero  $D(t)$  is considered afterwards.

### 3.1.2 The State Transition Matrix

The state transition matrix is introduced below which concerns the linear differential equation (1).

**Lemma 1:** *The equation (1) has the solution*

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, s)B(s)w(s)ds, \quad (3)$$

where the state transition matrix,  $\Phi(t, t_0)$ , satisfies

$$\dot{\Phi}(t, t_0) = \frac{d\Phi(t, t_0)}{dt} = A(t)\Phi(t, t_0), \quad (4)$$

with boundary condition

$$\Phi(t, t) = I. \quad (5)$$

**Proof:** Differentiating both sides of (3) and using Leibnitz's rule, that is,

$$\frac{\partial}{\partial t} \int_{\alpha(t)}^{\beta(t)} f(t, \tau) d\tau = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(t, \tau)}{\partial t} d\tau - f(t, \alpha) \frac{d\alpha(t)}{dt} + f(t, \beta) \frac{d\beta(t)}{dt}, \text{ gives}$$

---

"Life is good for only two things, discovering mathematics and teaching mathematics." *Siméon Denis Poisson*



$$\dot{x}(t) = \dot{\Phi}(t, t_0)x(t_0) + \int_{t_0}^t \dot{\Phi}(t, \tau)B(\tau)w(\tau)d\tau + \Phi(t, t)B(t)w(t). \quad (6)$$

Substituting (4) and (5) into the right-hand-side of (6) results in

$$\dot{x}(t) = A(t)\left(\Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)w(\tau)d\tau\right) + B(t)w(t). \quad (7)$$

### 3.1.3 The Lyapunov Differential Equation

The mathematical expectation,  $E\{x(t)x^T(\tau)\}$  of  $x(t)x^T(\tau)$ , is required below, which is defined as

$$E\{x(t)x^T(\tau)\} = \int_{-\infty}^{\infty} x(t)x^T(\tau)f_{xx}(x(t)x^T(\tau))dx(t), \quad (8)$$

where  $f_{xx}(x(t)x^T(\tau))$  is the probability density function of  $x(t)x^T(\tau)$ . A useful property of expectations is demonstrated in the following example.

**Example 1.** Suppose that  $x(t)$  is a stochastic random variable and  $h(t)$  is a continuous function, then

$$E\left\{\int_a^b h(t)x(t)x^T(\tau)dt\right\} = \int_a^b h(t)E\{x(t)x^T(\tau)\}dt. \quad (9)$$

To verify this, expand the left-hand-side of (9) to give

$$\begin{aligned} E\left\{\int_a^b h(t)x(t)x^T(\tau)dt\right\} &= \int_{-\infty}^{\infty} \int_a^b h(t)x(t)x^T(\tau)dt f_{xx}(x(t)x^T(\tau))dx(t) \\ &= \int_{-\infty}^{\infty} \int_a^b h(t)x(t)x^T(\tau)f_{xx}(x(t)x^T(\tau))dtdx(t). \end{aligned} \quad (10)$$

Using Fubini's theorem, that is,  $\int_c^d \int_a^b g(x, y)dx dy = \int_a^b \int_c^d g(x, y)dy dx$ , within (10) results in

$$E\left\{\int_a^b h(t)x(t)x^T(\tau)dt\right\} = \int_a^b \int_{-\infty}^{\infty} h(t)x(t)x^T(\tau)f_{xx}(x(t)x^T(\tau))dx(t)dt$$

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"It is a mathematical fact that the casting of this pebble from my hand alters the centre of gravity of the universe." *Thomas Carlyle*

$$= \int_a^b h(t) \int_{-\infty}^{\infty} x(t) x^T(\tau) f_{xx}(x(t) x^T(\tau)) dx(t) dt . \quad (11)$$

The result (9) follows from the definition (8) within (11).

The Dirac delta function,  $\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$ , satisfies the identity  $\int_{-\infty}^{\infty} \delta(t) dt = 1$ . In the foregoing development, use is made of the partitioning

$$\int_{-\infty}^0 \delta(t) dt = \int_0^{\infty} \delta(t) dt = 0.5 . \quad (12)$$

**Lemma 2:** In respect of equation (1), assume that  $w(t)$  is a zero-mean white process with  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t - \tau)$  that is uncorrelated with  $x(t_0)$ , namely,  $E\{w(t)x^T(t_0)\} = 0$ . Then the covariances  $P(t, \tau) = E\{x(t)x^T(\tau)\}$  and  $\dot{P}(t, \tau) = \frac{d}{dt} E\{x(t)x^T(\tau)\}$  satisfy the Lyapunov differential equation

$$\dot{P}(t, \tau) = A(t)P(t, \tau) + P(t, \tau)A^T(\tau) + B(t)Q(t)B^T(t) . \quad (13)$$

**Proof:** Using (1) within  $\frac{d}{dt} E\{x(t)x^T(\tau)\} = E\{\dot{x}(t)x^T(\tau) + x(t)\dot{x}^T(\tau)\}$  yields

$$\begin{aligned} \dot{P}(t, \tau) &= E\{A(t)x(t)x^T(\tau) + B(t)w(t)x^T(\tau)\} \\ &\quad + E\{x(t)x^T(\tau)A^T(\tau) + x(t)w^T(\tau)B^T(\tau)\} \\ &= A(t)P(t, \tau) + P(t, \tau)A^T(\tau) \\ &\quad + E\{B(t)w(t)x^T(\tau)\} + E\{x(t)w^T(\tau)B^T(\tau)\} . \end{aligned} \quad (14)$$

It follows from (1) and (3) that

$$\begin{aligned} E\{B(t)w(t)x^T(\tau)\} &= B(t)E\{w(t)x^T(0)\Phi(t, 0)\} \\ &\quad + B(t)E\left\{\int_{t_0}^t w(t)w^T(\tau)B^T(\tau)\Phi(t, \tau)d\tau\right\} \\ &= B(t)E\{w(t)x^T(0)\Phi(t, 0)\} \\ &\quad + B(t)\int_{t_0}^t E\{w(t)w^T(\tau)\}B^T(\tau)\Phi(t, \tau)d\tau . \end{aligned} \quad (15)$$

The assumptions  $E\{w(t)x^T(t_0)\} = 0$  and  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t - \tau)$  together with (15) lead to

$$\begin{aligned} E\{(B(t)w(t)x^T(\tau))\} &= B(t)Q(t)\int_{t_0}^t \delta(t - \tau)B^T(\tau)\Phi(t, \tau)d\tau \\ &= 0.5B(t)Q(t)B^T(t). \end{aligned} \quad (16)$$

The above Lyapunov differential equation follows by substituting (16) into (14).

In the case  $\tau = t$ , denote  $P(t, t) = E\{x(t)x^T(t)\}$  and  $\dot{P}(t, t) = \frac{d}{dt}E\{x(t)x^T(t)\}$ . Then the corresponding Lyapunov differential equation is written as

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + B(t)Q(t)B^T(t). \quad (17)$$

### 3.1.4 Conditional Expectations

The minimum-variance filter derivation that follows employs a conditional expectation formula, which is set out as follows. Consider a stochastic vector  $[x^T(t) \ y^T(t)]^T$  having means and covariances

$$E\left\{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right\} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \quad (18)$$

and

$$E\left\{\begin{bmatrix} x(t) - \bar{x} \\ y(t) - \bar{y} \end{bmatrix} \begin{bmatrix} x^T(t) - \bar{x}^T & y^T(t) - \bar{y}^T \end{bmatrix}\right\} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}. \quad (19)$$

respectively, where  $\Sigma_{yx} = \Sigma_{xy}^T$ . Suppose that it is desired to obtain an estimate of  $x(t)$  given  $y(t)$ , denoted by  $E\{x(t) | y(t)\}$ , which minimises  $E\{(x(t) - E\{x(t) | y(t)\})(x(t) - E\{x(t) | y(t)\})^T\}$ . A standard approach (e.g., see [18]) is to assume that the solution for  $E\{x(t) | y(t)\}$  is affine to  $y(t)$ , namely,

$$E\{x(t) | y(t)\} = Ay(t) + b, \quad (20)$$

where  $A$  and  $b$  are unknowns to be found. It follows from (20) that

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“As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.” *Albert Einstein*

$$\begin{aligned} & E\{(x(t) - E\{x(t) | y(t)\})(x(t) - E\{x(t) | y(t)\})^T\} \\ &= E\{x(t)x^T(t) - x(t)y^T(t)A^T - x(t)b^T - Ay(t)x^T(t) \\ & \quad + Ay(t)y^T(t)A^T - Ay(t)b^T - bx^T(t) + by^T(t)A^T + bb^T\}. \end{aligned} \quad (21)$$

Substituting  $E\{x(t)x^T(t)\} = \bar{x}\bar{x}^T + \Sigma_{xx}$ ,  $E\{x(t)y^T(t)\} = \bar{x}\bar{y}^T + \Sigma_{xy}$ ,  $E\{y(t)x^T(t)\} = \bar{y}\bar{x}^T + \Sigma_{yx}$ ,  $E\{y(t)y^T(t)\} = \bar{y}\bar{y}^T + \Sigma_{yy}$  into (21) and completing the squares yields

$$\begin{aligned} & E\{(x(t) - E\{x(t) | y(t)\})(x(t) - E\{x(t) | y(t)\})^T\} \\ &= (\bar{x} - A\bar{y} - b)(\bar{x} - A\bar{y} - b)^T + \begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \begin{bmatrix} I \\ -A^T \end{bmatrix}. \end{aligned} \quad (22)$$

The second term on the right-hand-side of (22) can be rearranged as

$$\begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \begin{bmatrix} I \\ -A^T \end{bmatrix} = (A - \Sigma_{xy}\Sigma_{yy}^{-1})\Sigma_{yy}(A - \Sigma_{xy}\Sigma_{yy}^{-1})^T + \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}.$$

Thus, the choice  $A = \Sigma_{xy}\Sigma_{yy}^{-1}$  and  $b = \bar{x} - A\bar{y}$  minimises (22), which gives

$$E\{x(t) | y(t)\} = \bar{x} + \Sigma_{xy}\Sigma_{yy}^{-1}(y(t) - \bar{y}) \quad (23)$$

and

$$E\{(x(t) - E\{x(t) | y(t)\})(x(t) - E\{x(t) | y(t)\})^T\} = \Sigma_{xx} + \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}. \quad (24)$$

The conditional mean estimate (23) is also known as the linear least mean square estimate [18]. An important property of the conditional mean estimate is established below.

**Lemma 3 (Orthogonal projections):** *In respect of the conditional mean estimate (23), in which the mean and covariances are respectively defined in (18) and (19), the error vector*

$$\tilde{x}(t) = x(t) - E\{x(t) | y(t)\}. \quad (25)$$

*is orthogonal to  $y(t)$ , that is,  $E\{\tilde{x}(t)y^T(t)\} = 0$ .*

**Proof [8],[18]:** From (23) and (25), it can be seen that

$$\begin{aligned}
E\{(\tilde{x}(t) - E\{\tilde{x}(t)\})(y(t) - E\{y(t)\})^T\} &= E\left\{\left(x(t) - \bar{x} - \Sigma_{xy}\Sigma_{yy}^{-1}(y(t) - \bar{y})\right)(y(t) - \bar{y})^T\right\} \\
&= \Sigma_{xy} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yy} \\
&= 0.
\end{aligned}$$

Sufficient background material has now been introduced for the finite-horizon filter (for time-varying systems) to be derived.

## 3.2 The Continuous-time Minimum-Variance Filter

### 3.2.1 Derivation of the Optimal Filter

Consider again the linear time-varying system  $\mathcal{G}: \mathbb{R}^m \rightarrow \mathbb{R}^p$  having the state-space realisation

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (26)$$

$$y(t) = C(t)x(t), \quad (27)$$

where  $A(t)$ ,  $B(t)$ ,  $C(t)$  are of appropriate dimensions and  $w(t)$  is a white process with

$$E\{w(t)\} = 0, \quad E\{w(t)w^T(\tau)\} = Q(t)\delta(t - \tau). \quad (28)$$

Suppose that observations

$$z(t) = y(t) + v(t) \quad (29)$$

are available, where  $v(t) \in \mathbb{R}^p$  is a white measurement noise process with

$$E\{v(t)\} = 0, \quad E\{v(t)v^T(\tau)\} = R(t)\delta(t - \tau) \quad (30)$$

and

$$E\{w(t)v^T(\tau)\} = 0. \quad (31)$$

The objective is to design a linear system  $\mathcal{H}$  that operates on the measurements  $z(t)$  and produces an estimate  $\hat{y}(t|t) = C(t)\hat{x}(t|t)$  of  $y(t) = C(t)x(t)$  given measurements at time  $t$ , so that the covariance  $E\{\tilde{y}(t|t)\tilde{y}^T(t|t)\}$  is minimised,

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“I have hardly known a mathematician who was capable of reasoning”. *Plato*

where  $\tilde{y}(t|t) = y(t) - \hat{y}(t|t)$  is the output estimation error. This output estimation problem is depicted in Fig. 2.

It is desired that  $\hat{x}(t|t)$  and the estimate  $\dot{\hat{x}}(t|t)$  of  $\dot{x}(t)$  are unbiased, namely

$$E\{x(t) - \hat{x}(t|t)\} = 0, \quad (32)$$

$$E\{\dot{x}(t|t) - \dot{\hat{x}}(t|t)\} = 0. \quad (33)$$

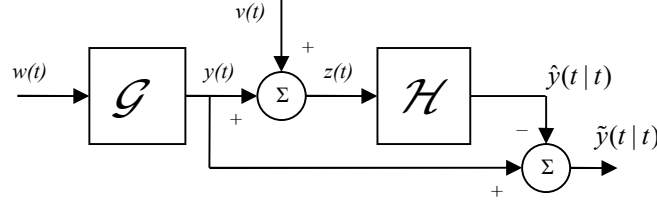


Fig. 2. The continuous-time output estimation problem. The objective is to find an estimate  $\hat{y}(t|t)$  of  $y(t)$  which minimises  $E\{(y(t) - \hat{y}(t|t))(y(t) - \hat{y}(t|t))^T\}$ .

If  $\hat{x}(t|t)$  is a conditional mean estimate, from Lemma 3, criterion (32) will be met. Criterion (33) can be satisfied if it is additionally assumed that  $E\{\dot{\hat{x}}(t|t)\} = A(t)\hat{x}(t|t)$ , since this yields  $E\{\dot{x}(t|t) - \dot{\hat{x}}(t|t)\} = A(t)(E\{x(t) - \hat{x}(t|t)\}) = 0$ . Thus, substituting  $E\left\{\begin{bmatrix} \dot{\hat{x}}(t|t) \\ z(t) \end{bmatrix}\right\} = \begin{bmatrix} A(t)\hat{x}(t|t) \\ C(t)\hat{x}(t|t) \end{bmatrix}$  into (23), yields the conditional mean estimate

$$\begin{aligned} \dot{\hat{x}}(t|t) &= A(t)\hat{x}(t|t) + K(t)(z(t) - C(t)\hat{x}(t|t)) \\ &= (A(t) - K(t)C(t))\hat{x}(t|t) + K(t)z(t), \end{aligned} \quad (34)$$

where  $K(t) = E\{x(t)z^T(t)\}E\{z(t)z^T(t)\}^{-1}$ . Equation (34) is known as the continuous-time Kalman filter (or the Kalman-Bucy filter) and is depicted in Fig. 3. This filter employs the state matrix  $A(t)$  akin to the signal generating model  $\mathcal{G}$ , which Kalman and Bucy call the message process [4]. The matrix  $K(t)$  is known as the filter gain, which operates on the error residual, namely the difference between the measurement  $z(t)$  and the estimated output  $C(t)\hat{x}(t)$ . The calculation of an optimal gain is addressed in the next section.

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“Art has a double face, of expression and illusion, just like science has a double face: the reality of error and the phantom of truth.” *René Daumal*

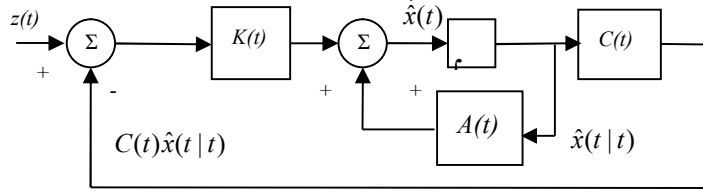


Fig. 3. The continuous-time Kalman filter which is also known as the Kalman-Bucy filter. The filter calculates conditional mean estimates  $\hat{x}(t|t)$  from the measurements  $z(t)$ .

### 3.2.2 The Riccati Differential Equation

Denote the state estimation error by  $\tilde{x}(t|t) = x(t) - \hat{x}(t|t)$ . It is shown below that the filter minimises the error covariance  $E\{\tilde{x}(t|t)\tilde{x}^T(t|t)\}$  if the gain is calculated as

$$K(t) = P(t)C^T(t)R^{-1}(t), \quad (35)$$

in which  $P(t) = E\{\tilde{x}(t|t)\tilde{x}^T(t|t)\}$  is the solution of the Riccati differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t) + B(t)Q(t)B^T(t). \quad (36)$$

**Lemma 4:** In respect of the state estimation problem defined by (26) - (31), suppose that there exists a solution

$$P(t) = P^T(t) \geq 0 \quad (37)$$

for the algebraic Riccati equation (36) satisfying

$$A(t) - P(t)C^T(t) \geq 0 \quad (38)$$

for all  $t$  in the interval  $[0, T]$ . Then the filter (34) having the gain (35) minimises  $P(t) = E\{\tilde{x}(t|t)\tilde{x}^T(t|t)\}$ .

**Proof:** Subtracting (34) from (26) results in

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“Somewhere, something incredible is waiting to be known.” *Carl Edward Sagan*

$$\dot{\tilde{x}}(t|t) = (A(t) - K(t)C(t))\tilde{x}(t|t) + B(t)w(t) - K(t)v(t). \quad (39)$$

Applying Lemma 2 to the error system (39) gives

$$\begin{aligned} \dot{P}(t) = & (A(t) - K(t)C(t))P(t) + P(t)(A(t) - K(t)C(t))^T \\ & + K(t)R(t)K^T(t) + B(t)Q(t)B^T(t) \end{aligned} \quad (40)$$

which can be rearranged as

$$\begin{aligned} \dot{P}(t) = & A(t)P(t) + P(t)A^T(t) + B(t)Q(t)B^T(t) \\ & + (K(t) - P(t)C^T(t)R^{-1}(t))R(t)(K^T(t) - R^{-1}(t)C^T(t)P(t)) \\ & + P(t)C^T(t)R^{-1}(t)C(t)P(t). \end{aligned} \quad (41)$$

Setting  $\dot{P}(t)$  equal to the zero matrix results in a stationary point at (35) which leads to (40). From the differential of (40)

$$\begin{aligned} \ddot{P}(t) = & (A(t) - P(t)C^T(t)R^{-1}(t)C(t))\dot{P}(t) \\ & + \dot{P}(t)(A^T(t) - P(t)C^T(t)R^{-1}(t)C(t)) \end{aligned} \quad (42)$$

and it can be seen that  $\ddot{P}(t) \geq 0$  provided that the assumptions (37) - (38) hold. Therefore,  $P(t) = E\{\tilde{x}(t|t)\tilde{x}^T(t|t)\}$  is minimised at (35).

The above development is somewhat brief and not very rigorous. Further discussions appear in [4] – [17]. It is tendered to show that the Kalman filter minimises the error covariance, provided of course that the problem assumptions are correct. In the case that it is desired to estimate an arbitrary linear combination  $C_1(t)$  of states, the optimal filter is given by the system

$$\dot{\hat{x}}(t|t) = A(t)\hat{x}(t|t) + K(t)(z(t) - C(t)\hat{x}(t|t)), \quad (43)$$

$$\hat{y}_1(t) = C_1(t)\hat{x}(t). \quad (44)$$

This filter minimises the error covariance  $C_1(t)P(t)C_1^T(t)$ . The generalisation of the Kalman filter for problems possessing deterministic inputs, correlated noises, and a direct feed-through term is developed below.



### 3.2.3 Including Deterministic Inputs

Suppose that the signal model is described by

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) + \mu(t) \quad (45)$$

$$y(t) = C(t)x(t) + \pi(t), \quad (46)$$

where  $\mu(t)$  and  $\pi(t)$  are deterministic (or known) inputs. In this case, the filtered state estimate can be obtained by including the deterministic inputs as follows

$$\dot{\hat{x}}(t|t) = A(t)\hat{x}(t|t) + K(t)(z(t) - C(t)\hat{x}(t|t) - \pi(t)) + \mu(t) \quad (47)$$

$$\hat{y}(t) = C(t)\hat{x}(t) + \pi(t). \quad (48)$$

It is easily verified that subtracting (47) from (45) yields the error system (39) and therefore, the Kalman filter's differential Riccati equation remains unchanged.

**Example 2.** Suppose that an object is falling under the influence of a gravitational field and it is desired to estimate its position over  $[0, t]$  from noisy measurements. Denote the object's vertical position, velocity and acceleration by  $x(t)$ ,  $\dot{x}(t)$  and  $\ddot{x}(t)$ , respectively. Let  $g$  denote the gravitational constant. Then  $\ddot{x}(t) = -g$  implies  $\dot{x}(t) = \dot{x}(0) - gt$ , so the model may be written as

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \mu(t), \quad (49)$$

$$z(t) = C \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + v(t),$$

where  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is the state matrix,  $\mu(t) = \begin{bmatrix} \dot{x}(0) - gt \\ -g \end{bmatrix}$  is a deterministic input and  $C = [1 \ 0]$  is the output mapping. Thus, the Kalman filter has the form

$$\begin{bmatrix} \dot{\hat{x}}(t|t) \\ \dot{\hat{\dot{x}}}(t|t) \end{bmatrix} = A \begin{bmatrix} \hat{x}(t|t) \\ \hat{\dot{x}}(t|t) \end{bmatrix} + K \left( z(t) - C \begin{bmatrix} \hat{x}(t|t) \\ \hat{\dot{x}}(t|t) \end{bmatrix} \right) + \mu(t), \quad (50)$$

$$\hat{y}(t|t) = C(t)\hat{x}(t|t), \quad (51)$$

where the gain  $K$  is calculated from (35) and (36), in which  $BQB^T = 0$ .

---

"I am tired of all this thing called science here. We have spent millions in that sort of thing for the last few years, and it is time it should be stopped." *Simon Cameron*

### 3.2.4 Including Correlated Process and Measurement Noise

Suppose that the process and measurement noises are correlated, that is,

$$E \left\{ \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} w^T(\tau) & v^T(\tau) \end{bmatrix} \right\} = \begin{bmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{bmatrix} \delta(t - \tau). \quad (52)$$

The equation for calculating the optimal state estimate remains of the form (34), however, the differential Riccati equation and hence the filter gain are different. The generalisation of the optimal filter that takes into account (52) was published by Kalman in 1963 [5]. Kalman's approach was to first work out the corresponding discrete-time Riccati equation and then derive the continuous-time version.

The correlated noises can be accommodated by defining the signal model equivalently as

$$\dot{x}(t) = \bar{A}(t)x(t) + B(t)\bar{w}(t) + \mu(t), \quad (53)$$

where

$$\bar{A}(t) = A(t) - B(t)S(t)R^{-1}(t)C(t) \quad (54)$$

is a new state matrix,

$$\bar{w}(t) = w(t) - S(t)R^{-1}(t)v(t) \quad (55)$$

is a new stochastic input that is uncorrelated with  $v(t)$ , and

$$\mu(t) = B(t)S(t)R^{-1}(t)y(t) \quad (56)$$

is a deterministic signal. It can easily be verified that the system (53) with the parameters (54) – (56), has the structure (26) with  $E\{w(t)v^T(\tau)\} = 0$ . It is convenient to define

$$\begin{aligned} \bar{Q}(t)\delta(t - \tau) &= E\{\bar{w}(t)\bar{w}^T(\tau)\} \\ &= E\{w(t)w^T(\tau)\} - E\{w(t)v^T(\tau)R^{-1}(t)S^T(t) \\ &\quad - S(t)R^{-1}(t)E\{v(t)w^T(\tau) \\ &\quad + S(t)R^{-1}(t)E\{v(t)v^T(\tau)\}R^{-1}(t)S^T(t) \end{aligned}$$

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“These, Gentlemen, are the opinions upon which I base my facts.” *Winston Leonard Spencer-Churchill*

$$= \left( Q(t) - S(t)R^{-1}(t)S^T(t) \right) \delta(t - \tau). \quad (57)$$

The corresponding Riccati differential equation is obtained by substituting  $\bar{A}(t)$  for  $A(t)$  and  $\bar{Q}(t)$  for  $Q(t)$  within (36), namely,

$$\dot{P}(t) = \bar{A}(t)P(t) + P(t)\bar{A}^T(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t) + B(t)\bar{Q}(t)B^T(t). \quad (58)$$

This can be rearranged to give

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - K(t)(t)R^{-1}(t)K^T(t) + B(t)Q(t)B^T(t), \quad (59)$$

in which the gain is now calculated as

$$K(t) = \left( P(t)C^T(t) + B(t)S(t) \right) R^{-1}(t). \quad (60)$$

### 3.2.5 Including a Direct Feedthrough Matrix

The approach of the previous section can be used to address signal models that possess a direct feedthrough matrix, namely,

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (61)$$

$$y(t) = C(t)x(t) + D(t)w(t). \quad (62)$$

As before, the optimal state estimate is given by

$$\dot{\hat{x}}(t|t) = A(t)\hat{x}(t|t) + K(t)(z(t) - C(t)\hat{x}(t|t)), \quad (63)$$

where the gain is obtained by substituting  $S(t) = Q(t)D^T(t)$  into (60),

$$K(t) = \left( P(t)C^T(t) + B(t)Q(t)D^T(t) \right) R^{-1}(t), \quad (64)$$

in which  $P(t)$  is the solution of the Riccati differential equation

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“No human investigation can be called real science if it cannot be demonstrated mathematically.”  
Leonardo di ser Piero da Vinci

$$\begin{aligned}\dot{P}(t) = & \left( A(t) - B(t)Q(t)D^T(t)R^{-1}(t)C(t) \right) P(t) \\ & + P(t) \left( A(t) - B(t)Q(t)D^T(t)R^{-1}(t)C(t) \right)^T \\ & + B(t) \left( Q(t) - Q(t)D(t)R^{-1}(t)D^T(t)Q(t) \right) B^T(t).\end{aligned}$$

Note that the above Riccati equation simplifies to

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - K(t)(t)R^{-1}(t)K^T(t) + Q(t). \quad (65)$$

### 3.3 The Continuous-time Steady-State Minimum-Variance Filter

#### 3.3.1 Riccati Differential Equation Monotonicity

This section sets out the simplifications for the case where the signal model is stationary (or time-invariant). In this situation the structure of the Kalman filter is unchanged but the gain is fixed and can be pre-calculated. Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bw(t), \quad (66)$$

$$y(t) = Cx(t), \quad (67)$$

together with the observations

$$z(t) = y(t) + v(t), \quad (68)$$

assuming that  $\text{Re}\{\lambda_i(A)\} < 0$ ,  $E\{w(t)\} = 0$ ,  $E\{w(t)w^T(\tau)\} = Q$ ,  $E\{v(t)\} = 0$ ,  $E\{v(t)v^T(\tau)\} = R$  and  $E\{w(t)v^T(\tau)\} = 0$ . It follows from the approach of Section 3 that the Riccati differential equation for the corresponding Kalman filter is given by

$$\dot{P}(t) = AP(t) + P(t)A^T - P(t)C^T R^{-1}CP(t) + BQB^T. \quad (69)$$

It will be shown that the solution for  $P(t)$  monotonically approaches a steady-state asymptote, in which case the filter gain can be calculated before running the filter. The following result is required to establish that the solutions of the above Riccati differential equation are monotonic.

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“Today's scientists have substituted mathematics for experiments, and they wander off through equation after equation, and eventually build a structure which has no relation to reality.” *Nikola Tesla*

**Lemma 5 [11], [19], [20]:** Suppose that  $X(t)$  is a solution of the Lyapunov differential equation

$$\dot{X}(t) = AX(t) + X(t)A^T \quad (70)$$

over an interval  $t \in [0, T]$ . Then the existence of a solution  $X(t_0) \geq 0$  implies  $X(t) \geq 0$  for all  $t \in [0, T]$ .

**Proof:** Denote the transition matrix of  $\dot{x}(t) = -A(t)x(t)$  by  $\Phi^T(t, \tau)$ , for which  $\dot{\Phi}(t, \tau) = -A^T(t)\Phi(t, \tau)$  and  $\dot{\Phi}^T(t, \tau) = -\Phi^T(t, \tau)A(t)$ . Let  $P(t) = \Phi^T(t, \tau)X(t)\Phi(t, \tau)$ , then from (70)

$$\begin{aligned} 0 &= \Phi^T(t, \tau) \left( \dot{X}(t) - AX(t) - X(t)A^T \right) \Phi(t, \tau) \\ &= \dot{\Phi}^T(t, \tau)X(t)\Phi(t, \tau) + \Phi^T(t, \tau)\dot{X}(t)\Phi(t, \tau) + \Phi^T(t, \tau)X(t)\dot{\Phi}(t, \tau) \\ &= \dot{P}(t). \end{aligned}$$

Therefore, a solution  $X(t_0) \geq 0$  of (70) implies that  $X(t) \geq 0$  for all  $t \in [0, T]$ .

The monotonicity of Riccati differential equations has been studied by Bucy [6], Wonham [23], Poubelle *et al* [19] and Freiling [20]. The latter's simple proof is employed below.<sup>16</sup>

**Lemma 6 [19], [20]:** Suppose for a  $t \geq 0$  and a  $\delta_t > 0$  there exist solutions  $P(t) \geq 0$  and  $P(t + \delta_t) \geq 0$  of the Riccati differential equations

$$\dot{P}(t) = AP(t) + P(t)A^T - P(t)C^T R^{-1}CP(t) + BQB^T \quad (71)$$

and

$$\dot{P}(t + \delta_t) = AP(t + \delta_t) + P(t + \delta_t)A^T - P(t + \delta_t)C^T R^{-1}CP(t + \delta_t) + BQB^T, \quad (72)$$

respectively, such that  $P(t) - P(t + \delta_t) \geq 0$ . Then the sequence of matrices  $P(t)$  is monotonic nonincreasing, that is,

$$P(t) - P(t + \delta_t) \geq 0, \text{ for all } t \geq \delta_t. \quad (73)$$

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<sup>16</sup>“Mathematics is written for mathematicians.” *Nicholaus Copernicus*

**Proof:** The conditions of the Lemma are the initial step of an induction argument. For the induction step, denote  $\dot{P}(\delta_t) = \dot{P}(t) - \dot{P}(t + \delta_t)$ ,  $P(\delta_t) = P(t) - P(t + \delta_t)$  and  $\bar{A} = AP(t)C^T R^{-1}C - 0.5P(\delta_t)$ . Then

$$\begin{aligned}\dot{P}(\delta_t) &= AP(\delta_t) + P(\delta_t)A^T - P(t + \delta_t)C^T R^{-1}CP(t + \delta_t) + P(t)C^T R^{-1}CP(t) \\ &= \bar{A}P(\delta_t) + P(\delta_t)\bar{A}^T,\end{aligned}$$

which is of the form (70), and so the result (73) follows.

A monotonic nondecreasing case can be established similarly – see [20].

### 3.3.2 Observability

The continuous-time system (66) – (67) is termed completely observable if the initial states,  $x(t_0)$ , can be uniquely determined from the inputs and outputs,  $w(t)$  and  $y(t)$ , respectively, over an interval  $[0, T]$ . A simple test for observability is given by the following lemma.

**Lemma 7 [10], [21].** Suppose that  $A \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{p \times n}$ . The system is observable if and only if the observability matrix  $O \in \mathbb{R}^{np \times n}$  is of rank  $n$ , where

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}. \quad (74)$$

**Proof:** Recall from Chapter 2 that the solution of (66) is

$$x(t) = e^{At}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bw(\tau)d\tau. \quad (75)$$

Since the input signal  $w(t)$  within (66) is known, it suffices to consider the unforced system  $\dot{x}(t) = Ax(t)$  and  $y(t) = Cx(t)$ , that is,  $Bw(t) = 0$ , which leads to

$$y(t) = Ce^{At}x(t_0). \quad (76)$$

The exponential matrix is defined as

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“You can observe a lot by just watching.” Lawrence Peter (Yogi) Berra

$$\begin{aligned}
e^{At} &= I + At + \frac{A^2 t^2}{2} + \cdots + \frac{A^N t^N}{N!} \\
&= \sum_{k=0}^{N-1} \alpha_k(t) A^k,
\end{aligned} \tag{77}$$

where  $\alpha_k(t) = t^k / k!$ . Substituting (77) into (76) gives

$$\begin{aligned}
y(t) &= \sum_{k=0}^{N-1} \alpha_k(t) C A^k x(t_0) \\
&= \alpha_0(t) C x(t_0) + \alpha_1(t) C A x(t_0) + \cdots + \alpha_{N-1}(t) C A^{N-1} x(t_0). \\
&= [\alpha_0(t) \quad \alpha_1(t) \quad \cdots \quad \alpha_{N-1}(t)] \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix} x(t_0).
\end{aligned} \tag{78}$$

From the Cayley-Hamilton Theorem [22],

$$\text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

for all  $N \geq n$ . Therefore, we can take  $N = n$  within (78). Thus, equation (78) uniquely determines  $x(t_0)$  if and only if  $O$  has full rank  $n$ .

A system that does not satisfy the above criterion is said to be unobservable. An alternate proof for the above lemma is provided in [10]. If a signal model is not observable then a Kalman filter cannot estimate all the states from the measurements.

**Example 3.** The pair  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $C = [1 \quad 0]$  is expected to be unobservable because one of the two states appears as a system output whereas the other is

---

“Who will observe the observers ?” *Arthur Stanley Eddington*

hidden. By inspection, the rank of the observability matrix,  $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ , is 1.

Suppose instead that  $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , namely measurements of both states are

available. Since the observability matrix  $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$  is of rank 2, the pair  $(A, C)$  is observable, that is, the states can be uniquely reconstructed from the measurements.

### 3.3.3 The Algebraic Riccati Equation

Some pertinent facts concerning the Riccati differential equation (69) are:

- Its solutions correspond to the covariance of the state estimation error.
- From Lemma 6, if it is suitably initialised then its solutions will be monotonically nonincreasing.
- If the pair  $(A, C)$  is observable then the states can be uniquely determined from the outputs, in the limit as  $t$  approaches infinity, the Riccati differential equation will have a unique steady state solution.

**Lemma 8 [20], [23], [24]:** Suppose that  $\text{Re}\{\lambda_i(A)\} < 0$ , the pair  $(A, C)$  is observable, then the solution of the Riccati differential equation (69) satisfies

$$\lim_{t \rightarrow \infty} P(t) = P, \quad (79)$$

where  $P$  is the solution of the algebraic Riccati equation

$$0 = AP + PA^T - PC^T R^{-1} CP + BQB^T. \quad (80)$$

A proof that the solution  $P$  is in fact unique appears in [24]. A standard way for calculating solutions to (80) arises by finding an appropriate set of Schur vectors

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“Stand firm in your refusal to remain conscious during algebra. In real life, I assure you, there is no such thing as algebra.” *Francis Ann Lebowitz*



for the Hamiltonian matrix  $H = \begin{bmatrix} A & -C^T R^{-1} C \\ B Q B^T & -A^T \end{bmatrix}$ , see [25] and the Hamiltonian solver within *Matlab*<sup>TM</sup>.

**Example 4.** Suppose that  $A = -1$  and  $B = C = Q = R = 1$ , for which the solution of the algebraic Riccati equation (80) is  $P = 0.4121$ . Using Euler's integration method (see Chapter 1) with  $\delta_t = 0.01$  and  $P(0) = 1$ , the calculated solutions of the Riccati differential equation (69) are listed in Table 1. The data in the table demonstrate that the Riccati differential equation solution converges to the algebraic Riccati equation solution and  $\lim_{t \rightarrow \infty} \dot{P}(t) = 0$ .

$t$	$P(t)$	$\dot{P}(t)$
1	0.9800	-2.00
10	0.8316	-1.41
100	0.4419	$-8.13 \cdot 10^{-2}$
1000	0.4121	$-4.86 \cdot 10^{-13}$

Table 1. Solutions of (69) for Example 4.

The so-called infinite-horizon (or stationary) Kalman filter is obtained by substituting time-invariant state-space parameters into (34) - (35) to give

$$\dot{\hat{x}}(t|t) = (A - KC)\hat{x}(t|t) + Kz(t), \quad (81)$$

$$\hat{y}(t|t) = C\hat{x}(t|t), \quad (82)$$

where

$$K = PC^T R^{-1}, \quad (83)$$

in which  $P$  is calculated by solving the algebraic Riccati equation (80). The output estimation filter (81) – (82) has the transfer function

$$H_{OE}(s) = C(sI - A + KC)^{-1} K. \quad (84)$$

---

"Arithmetic is being able to count up to twenty without taking off your shoes." *Mickey Mouse*

**Example 5.** Suppose that a signal  $y(t) \in \mathbb{R}$  is generated by the system

$$y(t) = \left( \frac{b_m \frac{d^m}{dt^m} + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} + \dots + b_1 \frac{d}{dt} + b_0}{a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \frac{d}{dt} + a_0} \right) w(t).$$

This system's transfer function is

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0},$$

which can be realised in the controllable canonical form [10]

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \text{ and } C = [b_m \quad b_{m-1} \quad \dots \quad b_1 \quad b_0].$$

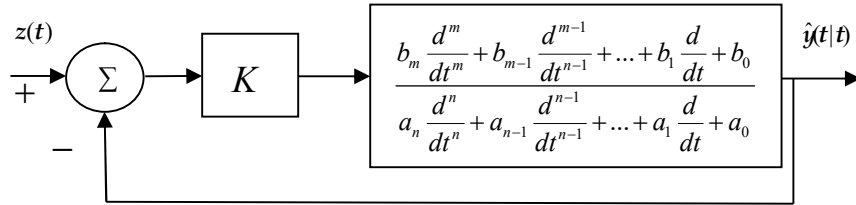


Fig. 4. The optimal filter for Example 5.

The optimal filter for estimating  $y(t)$  from noisy measurements (29) is obtained by using the above state-space parameters within (81) – (83). It has the structure depicted in Figs. 3 and 4. These figures illustrate two features of interest. First, the filter's model matches that within the signal generating process. Second, designing the filter is tantamount to finding an optimal gain.

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"If you think dogs can't count, try putting three dog biscuits in your pocket and then giving Fido two of them." *Phil Pastoret*

### 3.3.4 Equivalence of the Wiener and Kalman Filters

When the model parameters and noise statistics are time-invariant, the Kalman filter reverts to the Wiener filter. The equivalence of the Wiener and Kalman filters implies that spectral factorisation is the same as solving a Riccati equation. This observation is known as the Kalman-Yakubovich-Popov Lemma (or Positive Real Lemma) [15], [26], which assumes familiarity with the following Schur complement formula.

For any matrices  $\Phi_{11}$ ,  $\Phi_{12}$  and  $\Phi_{22}$ , where  $\Phi_{11}$  and  $\Phi_{22}$  are symmetric, the following are equivalent.

- (i)  $\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \geq 0$ .
- (ii)  $\Phi_{11} \geq 0$ ,  $\Phi_{22} \geq \Phi_{12}^T \Phi_{11}^{-1} \Phi_{12}$ .
- (iii)  $\Phi_{22} \geq 0$ ,  $\Phi_{11} \geq \Phi_{12} \Phi_{22}^{-1} \Phi_{12}^T$ .

The Kalman-Yakubovich-Popov Lemma is set out below. Further details appear in [15] and a historical perspective is provided in [26]. A proof of this Lemma makes use of the identity

$$-PA^T - AP = P(-sI - A^T) + (sI - A)P. \quad (85)$$

**Lemma 9 [15], [26]:** Consider the spectral density matrix

$$\Delta\Delta^H(s) = \begin{bmatrix} C(sI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} (-sI - A^T)^{-1} C^T \\ I \end{bmatrix}. \quad (86)$$

Then the following statements are equivalent:

- (i)  $\Delta\Delta^H(j\omega) \geq 0$  for all  $\omega \in (-\infty, \infty)$ .
- (ii)  $\begin{bmatrix} BQB^T + AP + PA^T & PC^T \\ CP & R \end{bmatrix} \geq 0$ .
- (iii) There exists a nonnegative solution  $P$  of the algebraic Riccati equation (80).

**Proof:** To establish equivalence between (i) and (iii), use (85) within (80) to obtain

$$P(-sI - A^T) + (sI - A)P = BQB^T - PC^T RCP. \quad (87)$$

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“Mathematics is the queen of sciences and arithmetic is the queen of mathematics.” *Carl Friedrich Gauss*

Premultiplying and postmultiplying (87) by  $C(sI - A)^{-1}$  and  $(-sI - A^T)^{-1}C^T$ , respectively, results in

$$\begin{aligned} C(sI - A)^{-1}PC^T + CP(-sI - A^T)^{-1}C^T \\ = C(sI - A)^{-1}(BQB^T - PC^TRCP)(-sI - A^T)^{-1}C^T. \end{aligned} \quad (88)$$

Hence,

$$\begin{aligned} \Delta\Delta(s) &= GQG(s) + R \\ &= C(sI - A)^{-1}BQB^T(-sI - A^T)^{-1}C^T + R \\ &\quad + C(sI - A)^{-1}PC^TRCP(-sI - A^T)^{-1}C^T \\ &= C(sI - A)^{-1}PC^TRCP(-sI - A^T)^{-1}C^T \\ &\quad + C(sI - A)^{-1}PC^T + CP(-sI - A^T)^{-1}C^T + R \\ &= \left( C(sI - A)^{-1}KR^{1/2} + R^{1/2} \right) \left( R^{1/2}K^T(-sI - A^T)^{-1}C^T + R^{1/2} \right) \\ &\geq 0. \end{aligned} \quad (89)$$

The Schur complement formula can be used to verify the equivalence of (ii) and (iii).

In Chapter 1, it is shown that the transfer function matrix of the optimal Wiener solution for output estimation is given by

$$H_{OE}(s) = I - R^{1/2}\Delta^{-1}(s), \quad (90)$$

where  $s = j\omega$  and

$$\Delta\Delta^H(s) = GQG^H(s) + R. \quad (91)$$

is the spectral density matrix of the measurements. It follows from (91) that

$$\Delta(s) = C(sI - A)^{-1}KR^{1/2} + R^{1/2}. \quad (92)$$

The Wiener filter (90) requires the spectral factor inverse,  $\Delta^{-1}(s)$ , which can be found from (92) and using  $[I + C(sI - A)^{-1}K]^{-1} = I + C(sI - A + KC)^{-1}K$  to obtain

$$\Delta^{-1}(s) = R^{-1/2} - R^{-1/2}C(sI - A + KC)^{-1}K. \quad (93)$$

Substituting (93) into (90) yields

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"Mathematics consists in proving the most obvious thing in the least obvious way." *George Polya*

$$H_{OE}(s) = C(sI - A + KC)^{-1}K, \quad (94)$$

which is identical to the minimum-variance output estimator (84).

**Example 6.** Consider a scalar output estimation problem where  $G(s) = (s + 1)^{-1}$ ,  $Q = 1$ ,  $R = 0.0001$  and the Wiener filter transfer function is

$$H(s) = 99(s + 100)^{-1}. \quad (95)$$

Applying the bilinear transform yields  $A = -1$ ,  $B = C = 1$ , for which the solution of (80) is  $P = 0.0099$ . By substituting  $K = PC^TR^{-1} = 99$  into (90), one obtains (95).

### 3.4 Chapter Summary

The Kalman-Bucy filter which produces state estimates  $\hat{x}(t|t)$  and output estimates  $\hat{y}(t|t)$  from the measurements  $z(t) = y(t) + v(t)$  at time  $t$  is summarised in Table 2. This filter minimises the variances of the state estimation error  $E\{(x(t) - \hat{x}(t|t))(x(t) - \hat{x}(t|t))^T\} = P(t)$  and the output estimation error  $E\{(y(t) - \hat{y}(t|t))(y(t) - \hat{y}(t|t))^T\} = C(t)P(t)C^T(t)$ .

	ASSUMPTIONS	MAIN RESULTS
Signals and system	$E\{w(t)\} = E\{v(t)\} = 0$ , $E\{w(t)w^T(t)\} = Q(t)$ and $E\{v(t)v^T(t)\} = R(t)$ are known. $A(t)$ , $B(t)$ and $C(t)$ are known.	$\dot{x}(t) = A(t)x(t) + B(t)w(t)$ $y(t) = C(t)x(t)$ $z(t) = y(t) + v(t)$
Filtered state and output		$\dot{\hat{x}}(t t) = A(t)\hat{x}(t t) + K(t)(z(t) - C(t)\hat{x}(t t))$ $\hat{y}(t t) = C(t)\hat{x}(t t)$
Filter gain and Riccati differential equation	$Q(t) > 0$ and $R(t) > 0$ .	$K(t) = P(t)C(t)R^{-1}(t)$ $\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t) + B(t)Q(t)B^T(t)$

Table 2. Main results for time-varying output estimation.

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“There are two ways to do great mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else - but persistent.” *Raoul Bott*

When the model parameters and noise covariances are time-invariant, the gain is also time-invariant and can be precalculated. The time-invariant filtering results are summarised in Table 3. In this stationary case, spectral factorisation is equivalent to solving a Riccati equation and the transfer function of the output estimation filter,  $H_{OE}(s) = C(sI - A + KC)^{-1}K$ , is identical to that of the Wiener filter. It is not surprising that the Wiener and Kalman filters are equivalent since they are both derived by completing the square of the error covariance.

	ASSUMPTIONS	MAIN RESULTS
Signals and system	$E\{w(t)\} = E\{v(t)\} = 0$ . $E\{w(t)w^T(t)\} = Q$ and $E\{v(t)v^T(t)\} = R$ are known. $A$ , $B$ and $C$ are known. The pair $(A, C)$ is observable.	$\dot{x}(t) = Ax(t) + Bw(t)$ $y(t) = Cx(t)$ $z(t) = y(t) + v(t)$
Filtered state and output		$\dot{\hat{x}}(t t) = A\hat{x}(t t) + K(z(t) - C\hat{x}(t t))$ $\hat{y}(t t) = C\hat{x}(t t)$
Filter gain and algebraic Riccati equation	$Q > 0$ and $R > 0$ .	$K = PCR^{-1}$ $0 = AP + PA^T - PC^T R^{-1}CP + BQB^T$

Table 3. Main results for time-invariant output estimation.

### 3.5 Problems

**Problem 1.** Show that  $\dot{x}(t) = A(t)x(t)$  has the solution  $x(t) = \Phi(t,0)x(0)$  where  $\dot{\Phi}(t,0) = A(t)\Phi(t,0)$  and  $\Phi(t,t) = I$ . Hint: use the approach of [13] and integrate both sides of  $\dot{x}(t) = A(t)x(t)$ .

**Problem 2.** Given that:

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“The scientific man does not aim at an immediate result. He does not expect that his advanced ideas will be readily taken up. His work is like that of the planter - for the future. His duty is to lay the foundation for those who are to come, and point the way.” *Nikola Tesla*

- (i) the Lyapunov differential equation for the system  $\dot{x}(t) = F(t)x(t) + G(t)w(t)$  is  $\frac{d}{dt}E\{x(t)x^T(t)\} = A(t)E\{x(t)x^T(t)\} + E\{x(t)x^T(t)\}F^T(t) + G(t)Q(t)G^T(t)$ ;
- (ii) the Kalman filter for the system  $\dot{x}(t) = A(t)x(t) + B(t)w(t)$ ,  $z(t) = C(t)x(t) + v(t)$  has the structure  $\dot{\hat{x}}(t|t) = A(t)\hat{x}(t|t) + K(t)(z(t) - C(t)\hat{x}(t|t))$ ;

write a Riccati differential equation for the evolution of the state error covariance and determine the optimal gain matrix  $K(t)$ .

**Problem 3.** Derive the Riccati differential equation for the model  $\dot{x}(t) = A(t)x(t) + B(t)w(t)$ ,  $z(t) = C(t)x(t) + v(t)$  with  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t - \tau)$ ,  $E\{v(t)v^T(\tau)\} = R(t)\delta(t - \tau)$  and  $E\{w(t)v^T(\tau)\} = S(t)\delta(t - \tau)$ . Hint: consider  $\dot{x}(t) = A(t)x(t) + B(t)w(t) + B(t)S(t)R^{-1}(t)(z(t) - C(t)x(t) - v(t))$ .

**Problem 4.** For output estimation problems with  $B = C = R = 1$ , calculate the algebraic Riccati equation solution, filter gain and transfer function for the following.

- |     |                         |     |                          |
|-----|-------------------------|-----|--------------------------|
| (a) | $A = -1$ and $Q = 8$ .  | (b) | $A = -2$ and $Q = 12$ .  |
| (c) | $A = -3$ and $Q = 16$ . | (d) | $A = -4$ and $Q = 20$ .  |
| (e) | $A = -5$ and $Q = 24$ . | (f) | $A = -6$ and $Q = 28$ .  |
| (g) | $A = -7$ and $Q = 32$ . | (h) | $A = -8$ and $Q = 36$ .  |
| (i) | $A = -9$ and $Q = 40$ . | (j) | $A = -10$ and $Q = 44$ . |

**Problem 5.** Prove the Kalman-Yakubovich-Popov Lemma for the case of

$$E\left\{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} w^T(\tau) & v^T(\tau) \end{bmatrix}\right\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta(t - \tau), \text{ i.e., show}$$

$$\Delta\Delta^H(s) = \begin{bmatrix} C(sI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S & R \end{bmatrix} \begin{bmatrix} (-sI - A^T)^{-1} C^T \\ I \end{bmatrix}.$$

**Problem 6.** Derive a state space formulation for minimum-mean-square-error equaliser using

$$\Delta^{-1}(s) = R^{-1/2} - R^{-1/2}C(sI - A + KC)^{-1}K.$$

## 3.6 Glossary

In addition to the terms listed in Chapter 1, the following have been used herein.

$\mathcal{G} : \mathbb{R}^p \rightarrow \mathbb{R}^q$	A linear system that operates on a $p$ -element input signal and produces a $q$ -element output signal.
$A(t), B(t),$	Time-varying state space matrices of appropriate dimension.

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“Mathematics is a game played according to certain simple rules with meaningless marks on paper.”  
David Hilbert

$C(t), D(t)$	The system $\mathcal{G}$ is assumed to have the realisation $\dot{x}(t) = A(t)x(t) + B(t)w(t)$ , $y(t) = C(t)x(t) + D(t)w(t)$ .
$Q(t)$ and $R(t)$	Covariance matrices of the nonstationary stochastic signals $w(t)$ and $v(t)$ , respectively.
$\Phi(t, 0)$	State transition matrix which satisfies $\dot{\Phi}(t, 0) = \frac{d\Phi(t, 0)}{dt} = A(t)\Phi(t, 0)$ with the boundary condition $\Phi(t, t) = I$ .
$\mathcal{G}^H$	Adjoint of $\mathcal{G}$ . The adjoint of a system having the state-space parameters $\{A(t), B(t), C(t), D(t)\}$ is a system parameterised by $\{-A^T(t), -C^T(t), B^T(t), D^T(t)\}$ .
$E\{.\}$ , $E\{x(t)\}$	Expectation operator, expected value of $x(t)$ .
$E\{x(t)   y(t)\}$	Conditional expectation, namely the estimate of $x(t)$ given $y(t)$ .
$\hat{x}(t   t)$	Conditional mean estimate of the state $x(t)$ given data at time $t$ .
$\tilde{x}(t   t)$	State estimation error which is defined by $\tilde{x}(t   t) = x(t) - \hat{x}(t   t)$ .
$K(t)$	Time-varying filter gain matrix.
$P(t)$	Time-varying error covariance, <i>i.e.</i> , $E\{\tilde{x}(t)\tilde{x}^T(t)\}$ , which is the solution of a Riccati differential equation.
$A, B, C, D$	Time-invariant state space matrices of appropriate dimension.
$Q$ and $R$	Time-invariant covariance matrices of the stationary stochastic signals $w(t)$ and $v(t)$ , respectively.
$O$	Observability matrix.
$SNR$	Signal to noise ratio.
$K$	Time-invariant filter gain matrix.
$P$	Time-invariant error covariance which is the solution of an algebraic Riccati equation.
$H$	Hamiltonian matrix.
$G(s)$	Transfer function matrix of the signal model.
$H(s)$	Transfer function matrix of the minimum-variance solution.
$H_{OE}(s)$	Transfer function matrix of the minimum-variance solution specialised for output estimation.

### 3.7 References

- [1] R. W. Bass, "Some reminiscences of control theory and system theory in the period 1955 – 1960: Introduction of Dr. Rudolf E. Kalman, *Real Time*, Spring/Summer Issue, The University of Alabama in Huntsville, 2002.
- [2] M. S. Grewal and A. P. Andrews, "Applications of Kalman Filtering in



- Areospace 1960 to the Present”, *IEEE Control Systems Magazine*, vol. 30, no. 3, pp. 69 – 78, June 2010.
- [3] R. E. Kalman, “A New Approach to Linear Filtering and Prediction Problems”, *Transactions of the ASME, Series D, Journal of Basic Engineering*, vol 82, pp. 35 – 45, 1960.
  - [4] R. E. Kalman and R. S. Bucy, “New results in linear filtering and prediction theory”, *Transactions of the ASME, Series D, Journal of Basic Engineering*, vol 83, pp. 95 – 107, 1961.
  - [5] R. E. Kalman, “New Methods in Wiener Filtering Theory”, *Proc. First Symposium on Engineering Applications of Random Function Theory and Probability*, Wiley, New York, pp. 270 – 388, 1963.
  - [6] R. S. Bucy, “Global Theory of the Riccati Equation”, *Journal of Computer and System Sciences*, vol. 1, pp. 349 – 361, 1967.
  - [7] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, Inc., New York, 1970.
  - [8] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York, 1971.
  - [9] A. Gelb, *Applied Optimal Estimation*, The Analytic Sciences Corporation, USA, 1974.
  - [10] T. Kailath, *Linear Systems*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1980.
  - [11] H. W. Knobloch and H. K. Kwakernaak, *Lineare Kontrolltheorie*, Springer-Verlag, Berlin, 1980.
  - [12] R. G. Brown and P. Y. C. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, John Wiley & Sons, Inc., USA, 1983.
  - [13] P. A. Ruymgaart and T. T. Soong, *Mathematics of Kalman-Bucy Filtering*, Second Edition, Springer-Verlag, Berlin, 1988.
  - [14] M. S. Grewal and A. P. Andrews, *Kalman Filtering, Theory and Practice*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1993.
  - [15] T. Kailath, A. H. Sayed and B. Hassibi, *Linear Estimation*, Prentice-Hall, Upper Saddle River, New Jersey, 2000.
  - [16] D. Simon, *Optimal State Estimation, Kalman  $H_\infty$  and Nonlinear Approaches*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.
  - [17] F. L. Lewis, L. Xie and D. Popa, *Optimal and Robust Estimation With an Introduction to Stochastic Control Theory*, Second Edition, CRC Press, Taylor & Francis Group, 2008.
  - [18] T. Söderström, *Discrete-time Stochastic Systems: estimation and control*, Springer-Verlag London Ltd., 2002.
  - [19] M. – A. Poubelle, R. R. Bitmead and M. R. Gevers, “Fake Algebraic Riccati Techniques and Stability”, *IEEE Transactions on Automatic Control*, vol. 33, no. 4, pp. 379 – 381, 1988.
  - [20] G. Freiling, V. Ionescu, H. Abou-Kandil and G. Jank, *Matrix Riccati*

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“But mathematics is the sister, as well as the servant, of the arts and is touched with the same madness and genius.” *Harold Marston Morse*

- Equations in Control and Systems Theory*, Birkhauser, Boston, 2003.
- [21] K. Ogata, *Matlab for Control Engineers*, Pearson Prentice Hall, Upper Saddle River, New Jersey, 2008.
  - [22] T. Kaczorek, “Cayley-Hamilton Theorem” in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer, 2001.
  - [23] W. M. Wonham, “On a Matrix Riccati Equation of Stochastic Control”, *SIAM Journal on Control*, vol. 6, no. 4, pp. 681 – 697, 1968.
  - [24] M. – A. Poubelle, I. R. Petersen, M. R. Gevers and R. R. Bitmead, “A Miscellany of Results on an Equation of Count J. F. Riccati”, *IEEE Transactions on Automatic Control*, vol. 31, no. 7, pp. 651 – 654, 1986.
  - [25] A. J. Laub, “A Schur Method for Solving Algebraic Riccati Equations”, *IEEE Transactions on Automatic Control*, vol. 24, no. 6, pp. 913 – 921, 1979.
  - [26] S. V. Gusev and A. L. Likhtarnikov, “Kalman-Popov-Yakubovich Lemma and the S-procedure: A Historical Essay”, *Automation and Remote Control*, vol. 67, no. 11, pp. 1768 – 1810, 2006.

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“Mathematicians are like Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different.” *Johann Wolfgang von Goethe*

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## 4. Discrete-Time, Minimum-Variance Filtering

### 4.1 Introduction

Kalman filters are employed wherever it is desired to recover data from the noise in an optimal way, such as in satellite orbit estimation, aircraft guidance, radar, communication systems, navigation, medical diagnosis and finance. Continuous-time problems that possess differential equations may be easier to describe in a state-space framework, however, the filters have higher implementation costs because an additional integration step and higher sampling rates are required. Conversely, although discrete-time state-space models may be less intuitive, the ensuing filter difference equations can be realised immediately.

The discrete-time Kalman filter calculates predicted states via the linear recursion

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k-1/k} + K_k (z_k - C_k \hat{x}_{k-1/k}),$$

where the predictor gain,  $K_k$ , is a function of the noise statistics and the model parameters. The above formula was reported by Rudolf E. Kalman in the 1960s [1], [2]. He has since received many awards and prizes, including the National Medal of Science, which was presented to him by President Barack Obama in 2009.

The Kalman filter calculations are simple and well-established. A possibly troublesome obstacle is expressing problems at hand within a state-space framework. This chapter derives the main discrete-time results to provide familiarity with state-space techniques and filter application. The continuous-time and discrete-time minimum-square-error Wiener filters were derived using a completing-the-square approach in Chapters 1 and 2, respectively. Similarly for time-varying continuous-time signal models, the derivation of the minimum-variance Kalman filter, presented in Chapter 3, relied on a least-mean-square (or conditional-mean) formula. This formula is used again in the solution of the discrete-time prediction and filtering problems. Predictions can be used when the measurements are irregularly spaced or missing at the cost of increased mean-square-error.

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“Man will occasionally stumble over the truth, but most of the time he will pick himself up and continue on.” *Winston Leonard Spencer-Churchill*

This chapter develops the prediction and filtering results for the case where the problem is nonstationary or time-varying. It is routinely assumed that the process and measurement noises are zero mean and uncorrelated. Nonzero mean cases can be accommodated by including deterministic inputs within the state prediction and filter output updates. Correlated noises can be handled by adding a term within the predictor gain and the underlying Riccati equation. The same approach is employed when the signal model possesses a direct-feedthrough term. A simplification of the generalised regulator problem from control theory is presented, from which the solutions of output estimation, input estimation (or equalisation), state estimation and mixed filtering problems follow immediately.

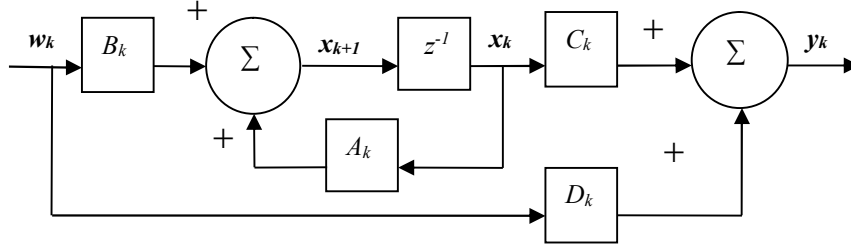


Fig. 1. The discrete-time system  $\mathcal{G}$  operates on the input signal  $w_k$  and produces the output  $y_k$ .

## 4.2 The Time-varying Signal Model

A discrete-time time-varying system  $\mathcal{G} : \mathbb{R}^{m \times N} \rightarrow \mathbb{R}^{p \times N}$  is assumed to have the state-space representation

$$x_{k+1} = A_k x_k + B_k w_k, \quad (1)$$

$$y_k = C_k x_k + D_k w_k, \quad (2)$$

where  $A_k \in \mathbb{R}^{n \times n}$ ,  $B_k \in \mathbb{R}^{n \times m}$ ,  $C_k \in \mathbb{R}^{p \times n}$  and  $D_k \in \mathbb{R}^{p \times m}$  over a finite interval  $k \in [1, N]$ . The  $w_k$  is a stochastic white process with

$$E\{w_k\} = 0, \quad E\{w_j w_k^T\} = Q_k \delta_{jk}, \quad (3)$$

in which  $\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$  is the Kronecker delta function. This system is

depicted in Fig. 1, in which  $z^{-1}$  is the unit delay operator. It is interesting to note that, at time  $k$  the current state

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“Rudy Kalman applied the state-space model to the filtering problem, basically the same problem discussed by Wiener. The results were astonishing. The solution was recursive, and the fact that the estimates could use only the past of the observations posed no difficulties.” *Jan. C. Willems*

$$x_k = A_{k-1}x_{k-1} + B_{k-1}w_{k-1}, \quad (4)$$

does not involve  $w_k$ . That is, unlike continuous-time systems, here there is a one-step delay between the input and output sequences. The simpler case of  $D_k = 0$ , namely,

$$y_k = C_k x_k, \quad (5)$$

is again considered prior to the inclusion of a nonzero  $D_k$ .

### 4.3 The State Prediction Problem

Suppose that noisy observations of (5) are available, that is,

$$z_k = y_k + v_k, \quad (6)$$

where  $v_k$  is a white measurement noise process with

$$E\{v_k\} = 0, \quad E\{v_j v_k^T\} = R_k \delta_{jk} \text{ and } E\{w_j v_k^T\} = 0. \quad (7)$$

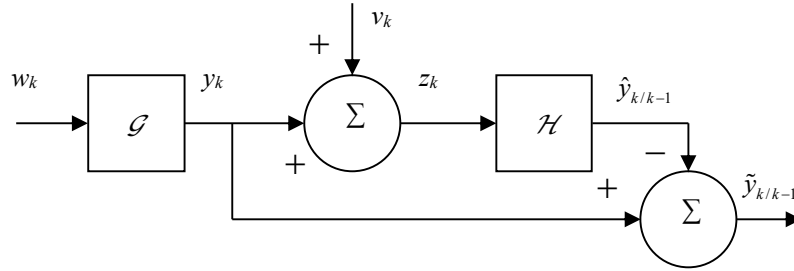


Fig. 2. The state prediction problem. The objective is to design a predictor  $\mathcal{H}$  which operates on the measurements and produces state estimates such that the variance of the error residual  $e_{k/k-1}$  is minimised.

It is noted above for the state recursion (4), there is a one-step delay between the current state and the input process. Similarly, it is expected that there will be one-step delay between the current state estimate and the input measurement. Consequently, it is customary to denote  $\hat{x}_{k/k-1}$  as the state estimate at time  $k$ , given measurements at time  $k-1$ . The  $\hat{x}_{k/k-1}$  is also known as the one-step-ahead state prediction. The objective here is to design a predictor  $\mathcal{H}$  that operates on the measurements  $z_k$  and produces an estimate,  $\hat{y}_{k/k-1} = C_k \hat{x}_{k/k-1}$ , of  $y_k = C_k x_k$ , so that

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“Prediction is very difficult, especially if it’s about the future.” *Niels Henrik David Bohr*

the covariance,  $E\{\tilde{y}_{k/k-1}\tilde{y}_{k/k-1}^T\}$ , of the error residual,  $\tilde{y}_{k/k-1} = y_k - \hat{y}_{k/k-1}$ , is minimised. This problem is depicted in Fig. 2

#### 4.4 The Discrete-time Conditional Mean Estimate

The predictor derivation that follows relies on the discrete-time version of the conditional-mean or least-mean-square estimate derived in Chapter 3, which is set out as follows. Consider a stochastic vector  $[\alpha_k^T \ \beta_k^T]^T$  having means and covariances

$$E\left\{\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}\right\} = \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix} \quad (8)$$

and

$$E\left\{\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}\begin{bmatrix} \alpha_k^T & \beta_k^T \end{bmatrix}\right\} = \begin{bmatrix} \Sigma_{\alpha_k\alpha_k} & \Sigma_{\alpha_k\beta_k} \\ \Sigma_{\beta_k\alpha_k} & \Sigma_{\beta_k\beta_k} \end{bmatrix}. \quad (9)$$

respectively, where  $\Sigma_{\beta_k\alpha_k} = \Sigma_{\alpha_k\beta_k}^T$ . An estimate of  $\alpha_k$  given  $\beta_k$ , denoted by  $E\{\alpha_k | \beta_k\}$ , which minimises  $E\{(\alpha_k - E\{\alpha_k | \beta_k\})(\alpha_k - E\{\alpha_k | \beta_k\})^T\}$ , is given by

$$E\{\alpha_k | \beta_k\} = \bar{\alpha} + \Sigma_{\alpha_k\beta_k} \Sigma_{\beta_k\beta_k}^{-1} (\beta_k - \bar{\beta}). \quad (10)$$

The above formula is developed in [3] and established for Gaussian distributions in [4]. A derivation is requested in the problems. If  $\alpha_k$  and  $\beta_k$  are scalars then (10) degenerates to the linear regression formula as is demonstrated below.

**Example 1 (Linear regression [5]).** The least-squares estimate  $\hat{\alpha}_k = a\beta_k + b$  of  $\alpha_k$  given data  $\alpha_k, \beta_k \in \mathbb{R}$  over  $[1, N]$ , can be found by minimising the performance objective  $J = \frac{1}{N} \sum_{k=1}^N (\alpha_k - \hat{\alpha})^2 = \frac{1}{N} \sum_{k=1}^N (\alpha_k - a\beta_k - b)^2$ . Setting  $\frac{dJ}{db} = 0$  yields  $b = \bar{\alpha} - a\bar{\beta}$ . Setting  $\frac{dJ}{da} = 0$ , substituting for  $b$  and using the definitions (8) – (9), results in  $a = \Sigma_{\alpha_k\beta_k} \Sigma_{\beta_k\beta_k}^{-1}$ .

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“I admired Bohr very much. We had long talks together, long talks in which Bohr did practically all the talking.” *Paul Adrien Maurice Dirac*

## 4.5 Minimum-Variance Prediction

It follows from (1), (6), together with the assumptions  $E\{w_k\} = 0$ ,  $E\{v_k\} = 0$ , that  $E\{x_{k+1}\} = E\{A_k x_k\}$  and  $E\{z_k\} = E\{C_k x_k\}$ . It is assumed that similar results hold in the case of predicted state estimates, that is,

$$E\left\{\begin{bmatrix} \hat{x}_{k+1} \\ z_k \end{bmatrix}\right\} = \begin{bmatrix} A_k \hat{x}_{k/k-1} \\ C_k \hat{x}_{k/k-1} \end{bmatrix}. \quad (11)$$

Substituting (11) into (10) and denoting  $\hat{x}_{k+1/k} = E\{\hat{x}_{k+1} | z_k\}$  yields the predicted state

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k-1} + K_k (z_k - C_k \hat{x}_{k/k-1}), \quad (12)$$

where  $K_k = E\{\hat{x}_{k+1} z_k^T\} E\{z_k z_k^T\}^{-1}$  is known as the predictor gain, which is designed in the next section. Thus, the optimal one-step-ahead predictor follows immediately from the least-mean-square (or conditional mean) formula. A more detailed derivation appears in [4]. The structure of the optimal predictor is shown in Fig. 3. It can be seen from the figure that  $\mathcal{H}$  produces estimates  $\hat{y}_{k/k-1} = C_k \hat{x}_{k/k-1}$  from the measurements  $z_k$ .

Let  $\tilde{x}_{k/k-1} = x_k - \hat{x}_{k/k-1}$  denote the state prediction error. It is shown below that the expectation of the prediction error is zero, that is, the predicted state estimate is unbiased.

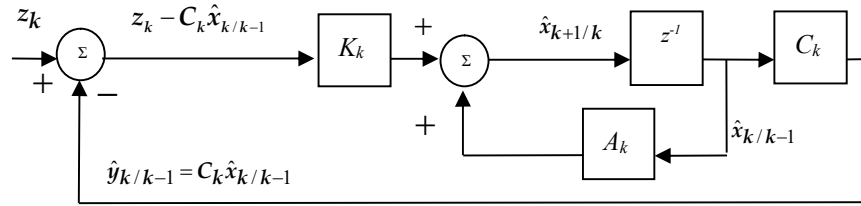


Fig. 3. The optimal one-step-ahead predictor which produces estimates  $\hat{x}_{k+1/k}$  of  $x_{k+1}$  given measurements  $z_k$ .

**Lemma 1:** Suppose that  $\hat{x}_{0/0} = x_0$ , then

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“When it comes to the future, there are three kinds of people: those who let it happen, those who make it happen, and those who wondered what happened.” *John M. Richardson Jr.*



$$E\{\tilde{x}_{k+1/k}\} = 0 \quad (13)$$

for all  $k \in [0, N]$ .

**Proof:** The condition  $\hat{x}_{0/0} = x_0$  is equivalent to  $\tilde{x}_{0/0} = 0$ , which is the initialisation step for an induction argument. Subtracting (12) from (1) gives

$$\tilde{x}_{k+1/k} = (A_k - K_k C_k) \tilde{x}_{k/k-1} + B_k w_k - K_k v_k \quad (14)$$

and therefore

$$E\{\tilde{x}_{k+1/k}\} = (A_k - K_k C_k) E\{\tilde{x}_{k/k-1}\} + B_k E\{w_k\} - K_k E\{v_k\}. \quad (15)$$

From assumptions (3) and (7), the last two terms of the right-hand-side of (15) are zero. Thus, (13) follows by induction.

## 4.6 Design of the Predictor Gain

It is shown below that the optimum predictor gain is that which minimises the prediction error covariance  $E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\}$ .

**Lemma 2:** In respect of the estimation problem defined by (1), (3), (5) - (7), suppose there exist solutions  $P_{k/k-1} = P_{k/k-1}^T \geq 0$  to the Riccati difference equation

$$P_{k+1/k} = A_k P_{k/k-1} A_k^T + B_k Q_k B_k^T - A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1} A_k^T, \quad (16)$$

over  $[0, N]$ , then the predictor gain

$$K_k = A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}, \quad (17)$$

within (12) minimises  $P_{k/k-1} = E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\}$ .

**Proof:** Constructing  $P_{k+1/k} = E\{\tilde{x}_{k+1/k} \tilde{x}_{k+1/k}^T\}$  using (3), (7), (14),  $E\{\tilde{x}_{k/k-1} w_k^T\} = 0$  and  $E\{\tilde{x}_{k/k-1} v_k^T\} = 0$  yields

$$P_{k+1/k} = (A_k - K_k C_k) P_{k/k-1} (A_k - K_k C_k)^T + B_k Q_k B_k^T + K_k R_k K_k^T, \quad (18)$$

which can be rearranged to give

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“To be creative you have to contribute something different from what you've done before. Your results need not be original to the world; few results truly meet that criterion. In fact, most results are built on the work of others.” *Lynne C. Levesque*

$$\begin{aligned}
 P_{k+1/k} = & A_k P_{k/k-1} A_k^T - A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1} A_k^T + B_k Q_k B_k^T \\
 & + (K_k - A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}) (C_k P_{k/k-1} C_k^T + R_k) \\
 & \times (K_k - A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1})^T,
 \end{aligned} \tag{19}$$

By inspection of (19), the predictor gain (17) minimises  $P_{k+1/k}$ .

## 4.7 Minimum-Variance Filtering

It can be seen from (12) that the predicted state estimate  $\hat{x}_{k/k-1}$  is calculated using the previous measurement  $z_{k-1}$  as opposed to the current data  $z_k$ . A state estimate, given the data at time  $k$ , which is known as the filtered state, can similarly be obtained using the linear least squares or conditional-mean formula. In Lemma 1 it was shown that the predicted state estimate is unbiased. Therefore, it is assumed that the expected value of the filtered state equals the expected value of the predicted state, namely,

$$E \left\{ \begin{bmatrix} \hat{x}_{k/k} \\ z_k \end{bmatrix} \right\} = \begin{bmatrix} \hat{x}_{k/k-1} \\ C_k \hat{x}_{k/k-1} \end{bmatrix}. \tag{20}$$

Substituting (20) into (10) and denoting  $\hat{x}_{k/k} = E\{\hat{x}_k | z_k\}$  yields the filtered estimate

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1}), \tag{21}$$

where  $L_k = E\{\hat{x}_k z_k^T\} E\{z_k z_k^T\}^{-1}$  is known as the filter gain, which is designed subsequently. Let  $\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}$  denote the filtered state error. It is shown below that the expectation of the filtered error is zero, that is, the filtered state estimate is unbiased.

**Lemma 3:** Suppose that  $\hat{x}_{0/0} = x_0$ , then

$$E\{\tilde{x}_{k/k}\} = 0 \tag{22}$$

for all  $k \in [0, N]$ .

**Proof:** Following the approach of [6], combining (4) - (6) results in  $z_k = C_k A_{k-1} x_{k-1} + C_k B_{k-1} w_{k-1} + v_k$ , which together with (21) yields

$$\tilde{x}_{k/k} = (I - L_k C_k) A_{k-1} \tilde{x}_{k-1/k-1} + (I - L_k C_k) B_{k-1} w_{k-1} - L_k v_k. \tag{23}$$

From (23) and the assumptions (3), (7), it follows that

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“A professor is one who can speak on any subject - for precisely fifty minutes.” *Norbert Wiener*

$$\begin{aligned} E\{\tilde{x}_{k/k}\} &= (I - L_k C_k) A_{k-1} E\{\tilde{x}_{k-1/k-1}\} \\ &= (I - L_k C_k) A_{k-1} \cdots (I - L_1 C_1) A_0 E\{\tilde{x}_{0/0}\}. \end{aligned} \quad (24)$$

Hence, with the initial condition  $\hat{x}_{0/0} = x_0$ ,  $E\{\tilde{x}_{k/k}\} = 0$ .

## 4.8 Design of the Filter Gain

It is shown below that the optimum filter gain is that which minimises the covariance  $E\{\tilde{x}_{k/k} \tilde{x}_{k/k}^T\}$ , where  $\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}$  is the filter error.

**Lemma 4:** In respect of the estimation problem defined by (1), (3), (5) - (7), suppose there exists a solution  $P_{k/k} = P_{k/k}^T \geq 0$  to the Riccati difference equation

$$P_{k/k} = P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1}, \quad (25)$$

over  $[0, N]$ , then the filter gain

$$L_k = P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}, \quad (26)$$

within (21) minimises  $P_{k/k} = E\{\tilde{x}_{k/k} \tilde{x}_{k/k}^T\}$ .

**Proof:** Subtracting  $\hat{x}_{k/k}$  from  $x_k$  yields  $\tilde{x}_{k/k} = x_k - \hat{x}_{k/k} = x_k - \hat{x}_{k/k-1} - L_k (C x_k + v_k - C \hat{x}_{k/k-1})$ , that is,

$$\tilde{x}_{k/k} = (I - L_k C_k) \tilde{x}_{k/k-1} - L_k v_k \quad (27)$$

and

$$P_{k/k} = (I - L_k C_k) P_{k/k-1} (I - L_k C_k)^T + L_k R_k L_k^T, \quad (28)$$

which can be rearranged as

$$\begin{aligned} P_{k/k} &= P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1} \\ &\quad + (L_k - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}) (C_k P_{k/k-1} C_k^T + R_k) \\ &\quad \times (L_k - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1})^T \end{aligned} \quad (29)$$

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“Before the advent of the Kalman filter, most mathematical work was based on Norbert Wiener's ideas, but the 'Wiener filtering' had proved difficult to apply. Kalman's approach, based on the use of state space techniques and a recursive least-squares algorithm, opened up many new theoretical and practical possibilities. The impact of Kalman filtering on all areas of applied mathematics, engineering, and sciences has been tremendous.” *Eduardo Daniel Sontag*

By inspection of (29), the filter gain (26) minimises  $P_{k/k}$ .

**Example 2 (Data Fusion).** Consider a filtering problem in which there are two measurements of the same state variable (possibly from different sensors), namely

$$A_k, B_k, Q_k \in \mathbb{R}, C_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } R_k = \begin{bmatrix} R_{1,k} & 0 \\ 0 & R_{2,k} \end{bmatrix}, \text{ with } R_{1,k}, R_{2,k} \in \mathbb{R}. \text{ Let } P_{k/k-1}$$

denote the solution of the Riccati difference equation (25). By applying Cramer's rule within (26) it can be found that the filter gain is given by

$$L_k = \begin{bmatrix} \frac{R_{2,k} P_{k/k-1}}{R_{2,k} P_{k/k-1} + R_{1,k} P_{k/k-1} + R_{1,k} R_{2,k}} & \frac{R_{1,k} P_{k/k-1}}{R_{2,k} P_{k/k-1} + R_{1,k} P_{k/k-1} + R_{1,k} R_{2,k}} \end{bmatrix},$$

from which it follows that  $\lim_{R_{1,k} \rightarrow 0, R_{2,k} \neq 0} L_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$  and  $\lim_{R_{2,k} \rightarrow 0, R_{1,k} \neq 0} L_k = \begin{bmatrix} 0 & 1 \end{bmatrix}$ . That is, when the first measurement is noise free, the filter ignores the second measurement and *vice versa*. Thus, the Kalman filter weights the data according to the prevailing measurement qualities.

## 4.9 The Predictor-Corrector Form

The Kalman filter may be written in the following predictor-corrector form. The corrected (or filtered) error covariances and states are respectively given by

$$\begin{aligned} P_{k/k} &= P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1} \\ &= P_{k/k-1} - L_k (C_k P_{k/k-1} C_k^T + R_k)^{-1} L_k^T \\ &= (I - L_k C_k) P_{k/k-1}, \end{aligned} \quad (30)$$

$$\begin{aligned} \hat{x}_{k/k} &= \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1}) \\ &= (I - L_k C_k) \hat{x}_{k/k-1} + L_k z_k, \end{aligned} \quad (31)$$

where  $L_k = P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}$ . Equation (31) is also known as the measurement update. The predicted state and error covariances are respectively given by

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k} = (A_k - K_k C_k) \hat{x}_{k/k-1} + K_k z_k \quad (32)$$

$$P_{k+1/k} = A_k P_{k/k} A_k^T + B_k Q_k B_k^T, \quad (33)$$

where  $K_k = A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}$ . It can be seen from (31) that the corrected estimate,  $\hat{x}_{k/k}$ , is obtained using measurements up to time k. This

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"I have been aware from the outset that the deep analysis of something which is now called Kalman filtering was of major importance. But even with this immodesty I did not quite anticipate all the reactions to this work." *Rudolf Emil Kalman*

contrasts with the prediction at time  $k + 1$  in (32), which is based on all previous measurements. The output estimate is given by

$$\begin{aligned}\hat{y}_{k/k} &= C_k \hat{x}_{k/k} \\ &= C_k \hat{x}_{k/k-1} + C_k L_k (z_k - C_k \hat{x}_{k/k-1}) \\ &= C_k (I - L_k C_k) \hat{x}_{k/k-1} + C_k L_k z_k.\end{aligned}\quad (34)$$

## 4.10 The *A Posteriori* Filter

The above predictor-corrector form is used in the construction of extended Kalman filters for nonlinear estimation problems (see Chapter 10). When state predictions are not explicitly required, the following one-line recursion for the filtered state can be employed. Substituting  $\hat{x}_{k/k-1} = A_{k-1} \hat{x}_{k-1/k-1}$  into  $\hat{x}_{k/k} = (I - L_k C_k) \hat{x}_{k/k-1} + L_k z_k$  yields  $\hat{x}_{k/k} = (I - L_k C_k) A_{k-1} \hat{x}_{k-1/k-1} + L_k z_k$ . Hence, the output estimator may be written as

$$\begin{bmatrix} \hat{x}_{k/k} \\ \hat{y}_{k/k} \end{bmatrix} = \begin{bmatrix} (I - L_k C_k) A_{k-1} & L_k \\ C_k & \end{bmatrix} \begin{bmatrix} \hat{x}_{k-1/k-1} \\ z_k \end{bmatrix}.\quad (35)$$

This form is called the *a posteriori* filter within [7], [8] and [9]. The absence of a direct feed-through matrix above reduces the complexity of the robust filter designs described in [7], [8] and [9].

## 4.11 The Information Form

Algebraically equivalent recursions of the Kalman filter can be obtained by propagating a so-called corrected information state

$$\underline{\hat{x}}_{k/k} = P_{k/k}^{-1} \hat{x}_{k/k},\quad (36)$$

and a predicted information state

$$\underline{\hat{x}}_{k+1/k} = P_{k+1/k}^{-1} \hat{x}_{k+1/k}.\quad (37)$$

The expression

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1},\quad (38)$$

which is variously known as the Matrix Inversion Lemma, the Sherman-Morrison formula and Woodbury's identity, is used to derive the information filter, see [3],

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"I have travelled the length and breadth of this country and talked with the best people, and I can assure you that data processing is a fad that won't last out the year." Editor in charge of business books for Prentice Hall, 1957.

[4], [11], [14] and [15]. To confirm the above identity, premultiply both sides of (38) by  $(A + BD^{-1}C)$  to obtain

$$\begin{aligned} I &= I + BCDA^{-1} - B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} - BCDA^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \\ &= I + BCDA^{-1} - B(I + CDA^{-1}B)^{-1}(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \\ &= I + BCDA^{-1} - BC(C^{-1} + DA^{-1}B)^{-1}(C^{-1} + DA^{-1}B)^{-1}DA^{-1}, \end{aligned}$$

from which the result follows. From the above Matrix Inversion Lemma and (30) it follows that

$$\begin{aligned} P_{k/k}^{-1} &= (P_{k/k-1} - P_{k/k-1}C_k^T(C_kP_{k/k-1}C_k^T + R_k)^{-1}C_kP_{k/k-1})^{-1} \\ &= P_{k/k-1}^{-1} + C_k^TR_k^{-1}C_k, \end{aligned} \quad (39)$$

assuming that  $P_{k/k-1}^{-1}$  and  $R_k^{-1}$  exist. An expression for  $P_{k+1/k}^{-1}$  can be obtained from the Matrix Inversion Lemma and (33), namely,

$$\begin{aligned} P_{k+1/k}^{-1} &= (A_kP_{k/k}A_k^T + B_kQ_kB_k^T)^{-1} \\ &= (F_k^{-1} + B_kQ_kB_k^T)^{-1}, \end{aligned} \quad (40)$$

where  $F_k = (A_kP_{k/k}A_k^T)^{-1} = A_k^{-T}P_{k/k}^{-1}A_k^{-1}$ , which gives

$$P_{k+1/k}^{-1} = (I - F_kB_k(B_k^TF_kB_k + Q_k^{-1})^{-1}B_k^T)F_k. \quad (41)$$

Another useful identity is

$$\begin{aligned} (A + BCD)^{-1}BC &= A^{-1}(I + BCDA^{-1})^{-1}BC \\ &= A^{-1}B(I + CDA^{-1}B)^{-1}C \\ &= A^{-1}B(C^{-1} + DA^{-1}B)^{-1}. \end{aligned} \quad (42)$$

From (42) and (39), the filter gain can be expressed as

$$\begin{aligned} L_k &= P_{k/k-1}C_k^T(C_kP_{k/k-1}C_k^T + R_k)^{-1} \\ &= (P_{k/k-1}^{-1} + C_k^TR_k^{-1}C_k)^{-1}C_k^TR_k^{-1} \\ &= P_{k/k}C_k^TR_k^{-1}. \end{aligned} \quad (43)$$

Premultiplying (39) by  $P_{k/k}$  and rearranging gives

$$I - L_kC_k = P_{k/k}P_{k/k-1}^{-1}. \quad (44)$$

It follows from (31), (36) and (44) that the corrected information state is given by

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“The fog of information can drive out knowledge.” *Daniel Joseph Boorstin*

$$\begin{aligned}
\underline{\hat{x}}_{k/k} &= P_{k/k}^{-1} \hat{x}_{k/k} \\
&= P_{k/k}^{-1} (I - L_k C_k) \hat{x}_{k/k-1} + P_{k/k}^{-1} L_k z_k \\
&= \hat{x}_{k/k-1} + C_k^T R_k^{-1} z_k.
\end{aligned} \tag{45}$$

The predicted information state follows from (37), (41) and the definition of  $F_k$ , namely,

$$\begin{aligned}
\underline{\hat{x}}_{k+1/k} &= P_{k+1/k}^{-1} \hat{x}_{k+1/k} \\
&= P_{k+1/k}^{-1} A_k \hat{x}_{k/k} \\
&= (I - F_k B_k (B_k^T F_k B_k + Q_k^{-1})^{-1} B_k^T) F_k A_k \hat{x}_{k/k} \\
&= (I - F_k B_k (B_k^T F_k B_k + Q_k^{-1})^{-1} B_k^T) A_k^{-T} \underline{\hat{x}}_{k/k}.
\end{aligned} \tag{46}$$

Recall from Lemma 1 and Lemma 3 that  $E\{x_k - \hat{x}_{k+1/k}\} = 0$  and  $E\{x_k - \hat{x}_{k/k}\} = 0$ , provided  $\hat{x}_{0/0} = x_0$ . Similarly, with  $\hat{x}_{0/0} = P_{0/0}^{-1} x_0$ , it follows that  $E\{x_k - P_{k+1/k} \underline{\hat{x}}_{k+1/k}\} = 0$  and  $E\{x_k - P_{k/k} \underline{\hat{x}}_{k/k}\} = 0$ . That is, the information states (scaled by the appropriate covariances) will be unbiased, provided that the filter is suitably initialised. The calculation cost and potential for numerical instability can influence decisions on whether to implement the predictor-corrector form (30) - (33) or the information form (39) - (46) of the Kalman filter. The filters have similar complexity, both require a  $p \times p$  matrix inverse in the measurement updates (31) and (45). However, inverting the measurement covariance matrix for the information filter may be troublesome when the measurement noise is negligible.

## 4.12 Comparison with Recursive Least Squares

The recursive least squares (RLS) algorithm is equivalent to the Kalman filter designed with the simplifications  $A_k = I$  and  $B_k = 0$ ; see the derivations within [10], [11]. For convenience, consider a more general RLS algorithm that retains the correct  $A_k$  but relies on the simplifying assumption  $B_k = 0$ . Under these conditions, denote the RLS algorithm's predictor gain by

$$\underline{K}_k = A_k \underline{P}_{k/k-1} C_k^T (C_k \underline{P}_{k/k-1} C_k^T + R_k)^{-1}, \tag{47}$$

where  $\underline{P}_{k/k-1}$  is obtained from the Riccati difference equation

$$\underline{P}_{k+1/k} = A_k \underline{P}_{k/k-1} A_k^T - A_k \underline{P}_{k/k-1} C_k^T (C_k \underline{P}_{k/k-1} C_k^T + R_k)^{-1} C_k \underline{P}_{k/k-1} A_k^T. \tag{48}$$

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"Information is the oxygen of the modern age. It seeps through the walls topped by barbed wire, it wafts across the electrified borders." *Ronald Wilson Reagan*

It is argued below that the cost of the above model simplification is an increase in mean-square-error.

**Lemma 5:** Let  $P_{k-1/k}$  denote the predicted error covariance within (33) for the optimal filter. Under the above conditions, the predicted error covariance,  $\bar{P}_{k/k-1}$ , of the RLS algorithm satisfies

$$P_{k/k-1} \leq \bar{P}_{k/k-1}. \quad (49)$$

**Proof:** From the approach of Lemma 2, the RLS algorithm's predicted error covariance is given by

$$\begin{aligned} \bar{P}_{k+1/k} = & A_k \bar{P}_{k/k-1} A_k^T - A_k \bar{P}_{k/k-1} C_k^T (C_k \bar{P}_{k/k-1} C_k^T + R_k)^{-1} C_k \bar{P}_{k/k-1} A_k^T + B_k Q_k B_k^T \\ & + (\underline{K}_k - A_k \bar{P}_{k/k-1} C_k^T (C_k \bar{P}_{k/k-1} C_k^T + R_k)^{-1}) (C_k \bar{P}_{k/k-1} C_k^T + R_k) \\ & \times (\underline{K}_k - A_k \bar{P}_{k/k-1} C_k^T (C_k \bar{P}_{k/k-1} C_k^T + R_k)^{-1})^T. \end{aligned} \quad (50)$$

The last term on the right-hand-side of (50) is nonzero since the above RLS algorithm relies on the erroneous assumption  $B_k Q_k B_k^T = 0$ . Therefore (49) follows.

### 4.13 Repeated Predictions

When there are gaps in the data record, or the data is irregularly spaced, state predictions can be calculated an arbitrary number of steps ahead. The one-step-ahead prediction is given by (32). The two, three and  $j$ -step-ahead predictions, given data at time  $k$ , are calculated as

$$\hat{x}_{k+2/k} = A_{k+1} \hat{x}_{k+1/k} \quad (51)$$

$$\hat{x}_{k+3/k} = A_{k+2} \hat{x}_{k+2/k} \quad (52)$$

$$\begin{aligned} & \vdots \\ \hat{x}_{k+j/k} &= A_{k+j-1} \hat{x}_{k+j-1/k}, \end{aligned} \quad (53)$$

see also [4], [12]. The corresponding predicted error covariances are given by

$$P_{k+2/k} = A_{k+1} P_{k+1/k} A_{k+1}^T + B_{k+1} Q_{k+1} B_{k+1}^T \quad (54)$$

$$P_{k+3/k} = A_{k+2} P_{k+2/k} A_{k+2}^T + B_{k+2} Q_{k+2} B_{k+2}^T \quad (55)$$

$$\begin{aligned} & \vdots \\ P_{k+j/k} &= A_{k+j-1} P_{k+j-1/k} A_{k+j-1}^T + B_{k+j-1} Q_{k+j-1} B_{k+j-1}^T. \end{aligned} \quad (56)$$

---

“All of the books in the world contain no more information than is broadcast as video in a single large American city in a single year. Not all bits have equal value.” *Carl Edward Sagan*



Another way to handle missing measurements at time  $i$  is to set  $C_i = 0$ , which leads to the same predicted states and error covariances. However, the cost of relying on repeated predictions is an increased mean-square-error which is demonstrated below.

**Lemma 6:**

- (i)  $P_{k/k} \leq P_{k/k-1}$ .
- (ii) Suppose that

$$A_k A_k^T + B_k Q_k B_k^T \geq I \quad (57)$$

for all  $k \in [0, N]$ , then  $P_{k+j/k} \geq P_{k+j-1/k}$  for all  $(j+k) \in [0, N]$ .

**Proof:**

- (i) The claim follows by inspection of (30) since  $L_{k-1}(C_{k-1}P_{k-1/k-2}C_{k-1}^T + R_{k-1})L_{k-1}^T \geq 0$ . Thus, the filter outperforms the one-step-ahead predictor.
- (ii) For  $P_{k+j-1/k} \geq 0$ , condition (57) yields  $A_{k+j-1}P_{k+j-1/k}A_{k+j-1}^T + B_{k+j-1}Q_{k+j-1}B_{k+j-1}^T \geq P_{k+j-1/k}$  which together with (56) results in  $P_{k+j/k} \geq P_{k+j-1/k}$ .

**Example 3.** Consider a filtering problem where  $A = 0.9$  and  $B = C = Q = R = 1$ , for which  $AA^T + BQB^T = 1.81 > 1$ . The predicted error covariances,  $P_{k+j/k}$ ,  $j = 1 \dots 10$ , are plotted in Fig. 4. The monotonically increasing sequence of error variances shown in the figure demonstrates that degraded performance occurs during repeated predictions. Fig. 5 shows some sample trajectories of the model output (dotted line), filter output (crosses) and predictions (circles) assuming that  $z_3 \dots z_8$  are unavailable. It can be seen from the figure that the prediction error increases with time  $k$ , which illustrates Lemma 6.

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“Where a calculator on the ENIAC is equipped with 18,000 vacuum tubes and weighs 30 tones, computers in the future may have only 1,000 vacuum tubes and perhaps weigh 1.5 tons.” *Popular Mechanics*, 1949

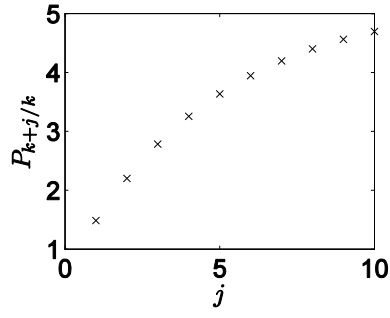


Fig. 4. Predicted error variances for Example 3.

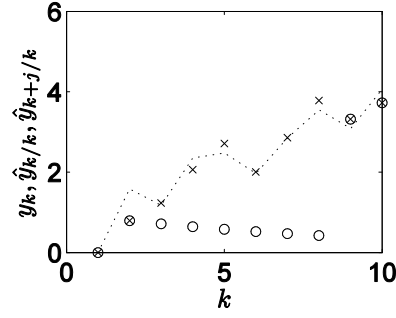


Fig. 5. Sample trajectories for Example 3:  $y_k$  (dotted line),  $\hat{y}_{k/k}$  (crosses) and  $\hat{y}_{k+j/k}$  (circles).

## 4.14 Accommodating Deterministic Inputs

Suppose that the signal model is described by

$$x_{k+1} = A_k x_k + B_k w_k + \mu_k, \quad (58)$$

$$y_k = C_k x_k + \pi_k, \quad (59)$$

where  $\mu_k$  and  $\pi_k$  are deterministic inputs (such as known non-zero means). The modifications to the Kalman recursions can be found by assuming  $\hat{x}_{k+1/k} = A_k \hat{x}_{k/k} + \mu_k$  and  $\hat{y}_{k/k-1} = C_k \hat{x}_{k/k-1} + \pi_k$ . The filtered and predicted states are then given by

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1} - \pi_k) \quad (60)$$

and

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k} + \mu_k \quad (61)$$

$$= A_k \hat{x}_{k/k-1} + K_k (z_k - C_k \hat{x}_{k/k-1} - \pi_k) + \mu_k, \quad (62)$$

respectively. Subtracting (62) from (58) gives

$$\begin{aligned} \tilde{x}_{k+1/k} &= A_k \tilde{x}_{k/k-1} - K_k (C_k \tilde{x}_{k/k-1} + \pi_k + v_k - \pi_k) + B_k w_k + \mu_k - \mu_k \\ &= (A_k - K_k C_k) \tilde{x}_{k/k-1} + B_k w_k - K_k v_k, \end{aligned} \quad (63)$$

where  $\tilde{x}_{k/k-1} = x_k - \hat{x}_{k/k-1}$ . Therefore, the predicted error covariance,

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“I think there is a world market for maybe five computers.” *Thomas John Watson*

$$\begin{aligned}
P_{k+1/k} &= (A_k - K_k C_k) P_{k/k-1} (A_k - K_k C_k)^T + B_k Q_k B_k^T + K_k R_k K_k^T \\
&= A_k P_{k/k-1} A_k^T - K_k (C_k P_{k/k-1} C_k^T + R_k) K_k^T + B_k Q_k B_k^T,
\end{aligned} \tag{64}$$

is unchanged. The filtered output is given by

$$\hat{y}_{k/k} = C_k \hat{x}_{k/k} + \pi_k. \tag{65}$$

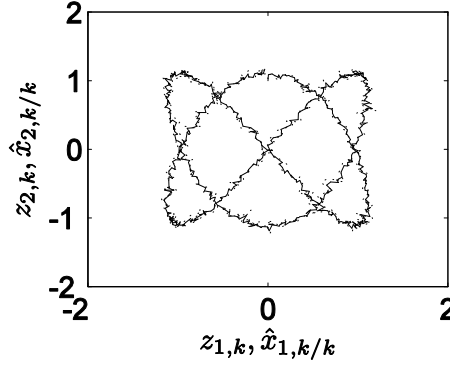


Fig. 6. Measurements (dotted line) and filtered states (solid line) for Example 4.

**Example 4.** Consider a filtering problem where  $A = \text{diag}(0.1, 0.1)$ ,  $B = C = \text{diag}(1, 1)$ ,  $Q = R = \text{diag}(0.001, 0.001)$ , with  $\mu_k = \begin{bmatrix} \sin(2k) \\ \cos(3k) \end{bmatrix}$ . The filtered states calculated from (60) are shown in Fig. 6. The resulting Lissajous figure illustrates that states having nonzero means can be modelled using deterministic inputs.

## 4.15 Correlated Process and Measurement Noises

Consider the case where the process and measurement noises are correlated

$$E \left\{ \begin{bmatrix} w_j \\ v_j \end{bmatrix} \begin{bmatrix} w_k^T & v_k^T \end{bmatrix} \right\} = \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \delta_{jk}. \tag{66}$$

The generalisation of the optimal filter that takes the above into account was published by Kalman in 1963 [2]. The expressions for the state prediction

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k-1} + K_k (z_k - C_k \hat{x}_{k/k-1}) \tag{67}$$

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“There is no reason anyone would want a computer in their home.” *Kenneth Harry Olson*

and the state prediction error

$$\tilde{x}_{k+1/k} = (A_k - K_k C_k) \tilde{x}_{k/k-1} + B_k w_k - K_k v_k \quad (68)$$

remain the same. It follows from (68) that

$$E\{\tilde{x}_{k+1/k}\} = (A_k - K_k C_k) E\{\tilde{x}_{k/k-1}\} + \begin{bmatrix} B_k & -K_k \end{bmatrix} \begin{bmatrix} E\{w_k\} \\ E\{v_k\} \end{bmatrix}. \quad (69)$$

As before, the optimum predictor gain is that which minimises the prediction error covariance  $E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\}$ .

**Lemma 7:** *In respect of the estimation problem defined by (1), (5), (6) with noise covariance (66), suppose there exist solutions  $P_{k/k-1} = P_{k/k-1}^T \geq 0$  to the Riccati difference equation*

$$P_{k+1/k} = A_k P_{k/k-1} A_k^T + B_k Q_k B_k^T - (A_k P_{k/k-1} C_k^T + B_k S_k)(C_k P_{k/k-1} C_k^T + R_k)^{-1} (A_k P_{k/k-1} C_k^T + B_k S_k)^T \quad (70)$$

over  $[0, N]$ , then the state prediction (67) with the gain

$$K_k = (A_k P_{k/k-1} C_k^T + B_k S_k)(C_k P_{k/k-1} C_k^T + R_k)^{-1}, \quad (71)$$

minimises  $P_{k/k-1} = E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\}$ .

**Proof:** It follows from (69) that

$$\begin{aligned} E\{\tilde{x}_{k+1/k} \tilde{x}_{k+1/k}^T\} &= (A_k - K_k C_k) E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\} (A_k - K_k C_k)^T \\ &\quad + \begin{bmatrix} B_k & -K_k \end{bmatrix} \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \begin{bmatrix} B_k^T \\ -K_k^T \end{bmatrix} \\ &= (A_k - K_k C_k) E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\} (A_k - K_k C_k)^T \\ &\quad + B_k Q_k B_k^T + K_k R_k K_k^T - B_k S_k K_k^T - K_k S_k B_k^T. \end{aligned} \quad (72)$$

Expanding (72) and denoting  $P_{k/k-1} = E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\}$  gives

$$\begin{aligned} P_{k+1/k} &= A_k P_{k/k-1} A_k^T + B_k Q_k B_k^T \\ &\quad - (A_k P_{k/k-1} C_k^T + B_k S_k)(C_k P_{k/k-1} C_k^T + R_k)^{-1} (A_k P_{k/k-1} C_k^T + B_k S_k)^T \\ &\quad + \left( K_k - (A_k P_{k/k-1} C_k^T + B_k S_k)(C_k P_{k/k-1} C_k^T + R_k)^{-1} \right) (C_k P_{k/k-1} C_k^T + R_k) \\ &\quad \times \left( K_k - (A_k P_{k/k-1} C_k^T + B_k S_k)(C_k P_{k/k-1} C_k^T + R_k)^{-1} \right)^T. \end{aligned} \quad (73)$$

By inspection of (73), the predictor gain (71) minimises  $P_{k+1/k}$ .

Thus, the predictor gain is calculated differently when  $w_k$  and  $v_k$  are correlated. The calculation of the filtered state and filtered error covariance are unchanged, viz.

$$\hat{x}_{k/k} = (I - L_k C_k) \hat{x}_{k/k-1} + L_k z_k, \quad (74)$$

$$P_{k/k} = (I - L_k C_k) P_{k/k-1} (I - L_k C_k)^T + L_k R_k L_k^T, \quad (75)$$

where

$$L_k = P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}. \quad (76)$$

However,  $P_{k/k-1}$  is now obtained from the Riccati difference equation (70).

## 4.16 Including a Direct-Feedthrough Matrix

Suppose now that the signal model possesses a direct-feedthrough matrix,  $D_k$ , namely

$$x_{k+1} = A_k x_k + B_k w_k, \quad (77)$$

$$y_k = C_k x_k + D_k w_k. \quad (78)$$

Let the observations be denoted by

$$z_k = C_k x_k + \underline{v}_k, \quad (79)$$

where  $\underline{v}_k = D_k w_k + v_k$ , under the assumptions (3) and (7). It follows that

$$E \left\{ \begin{bmatrix} w_j \\ \underline{v}_j \end{bmatrix} \begin{bmatrix} w_k^T & \underline{v}_k^T \end{bmatrix} \right\} = \begin{bmatrix} Q_k & Q_k D_k^T \\ D_k Q_k & D_k Q_k D_k^T + R_k \end{bmatrix} \delta_{jk}. \quad (80)$$

The approach of the previous section may be used to obtain the minimum-variance predictor for the above system. Using (80) within Lemma 7 yields the predictor gain

$$K_k = (A_k P_{k/k-1} C_k^T + B_k Q_k D_k^T) \Omega_k^{-1}, \quad (81)$$

where

$$\Omega_k = C_k P_{k/k-1} C_k^T + D_k Q_k D_k^T + R_k \quad (82)$$

---

“Everything that can be invented has been invented.” *Charles Holland Duell*

and  $P_{k/k-1}$  is the solution of the Riccati difference equation

$$P_{k+1/k} = A_k P_{k/k-1} A_k^T - K_k \Omega_k K_k^T + B_k Q_k B_k^T. \quad (83)$$

The filtered states can be calculated from (74), (82), (83) and  $L_k = P_{k/k-1} C_k^T \Omega_k^{-1}$ .

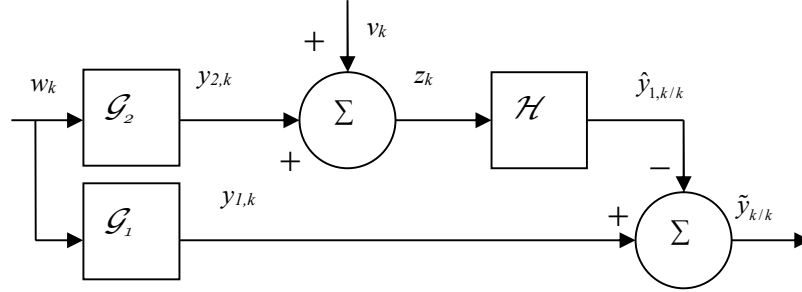


Fig. 7. The general filtering problem. The objective is to estimate the output of  $\mathcal{G}_1$  from noisy measurements of the output of  $\mathcal{G}_2$ .

## 4.17 Solution of the General Filtering Problem

The general filtering problem is shown in Fig. 7, in which it is desired to develop a filter  $\mathcal{H}$  that operates on noisy measurements of  $\mathcal{G}_2$  and estimates the output of  $\mathcal{G}_1$ . Frequency domain solutions for time-invariant systems were developed in Chapters 1 and 2. Here, for the time-varying case, it is assumed that the system  $\mathcal{G}_2$  has the state-space realisation

$$x_{k+1} = A_k x_k + B_k w_k, \quad (84)$$

$$y_{2,k} = C_{2,k} x_k + D_{2,k} w_k. \quad (85)$$

Suppose that the system  $\mathcal{G}_1$  has the realisation (84) and

$$y_{1,k} = C_{1,k} x_k + D_{1,k} w_k. \quad (86)$$

The objective is to produce estimates  $\hat{y}_{1,k/k}$  of  $y_{1,k}$  from the measurements

$$z_k = C_{2,k} x_k + v_k, \quad (87)$$

---

“This ‘telephone’ has too many shortcomings to be seriously considered as a means of communication. The device is inherently of no value to us.” *Western Union* memo, 1876

where  $\underline{v}_k = D_{2,k} w_k + v_k$ , so that the variance of the estimation error,

$$\tilde{y}_{k/k} = y_{1,k} - \hat{y}_{1,k/k}, \quad (88)$$

is minimised. The predicted state follows immediately from the results of the previous sections, namely,

$$\begin{aligned} \hat{x}_{k+1/k} &= A_k \hat{x}_{k/k-1} + K_k (z_k - C_{2,k} \hat{x}_{k/k-1}) \\ &= (A_k - K_k C_{2,k}) \hat{x}_{k/k-1} + K_k z_k \end{aligned} \quad (89)$$

where

$$K_k = (A_k P_{k/k-1} C_{2,k}^T + B_k Q_k D_{2,k}^T) \Omega_k^{-1} \quad (90)$$

and

$$\Omega_k = C_{2,k} P_{k/k-1} C_{2,k}^T + D_{2,k} Q_k D_{2,k}^T + R_k, \quad (91)$$

in which  $P_{k/k-1}$  evolves from

$$P_{k+1/k} = A_k P_{k/k-1} A_k^T - K_k \Omega_k K_k^T + B_k Q_k B_k^T. \quad (92)$$

In view of the structure (89), an output estimate of the form

$$\begin{aligned} \hat{y}_{1,k/k} &= C_{1,k} \hat{x}_{k/k-1} + L_k (z_k - C_{2,k} \hat{x}_{k/k-1}) \\ &= (C_{1,k} - L_k C_{2,k}) \hat{x}_{k/k-1} + L_k z_k, \end{aligned} \quad (93)$$

is sought, where  $L_k$  is a filter gain to be designed. Subtracting (93) from (86) gives

$$\begin{aligned} \tilde{y}_{k/k} &= y_{1,k} - \hat{y}_{1,k/k} \\ &= (C_{1,k} - L_k C_{2,k}) \tilde{x}_{k/k-1} + \begin{bmatrix} D_{1,k} & -L_k \end{bmatrix} \begin{bmatrix} w_k \\ \underline{v}_k \end{bmatrix}. \end{aligned} \quad (94)$$

It is shown below that an optimum filter gain can be found by minimising the output error covariance  $E\{\tilde{y}_{k/k} \tilde{y}_{k/k}^T\}$ .

**Lemma 8:** *In respect of the estimation problem defined by (84) - (88), the output estimate  $\hat{y}_{1,k/k}$  with the filter gain*

$$L_k = (C_{1,k} P_{k/k-1} C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T) \Omega_k^{-1} \quad (95)$$

---

“The wireless music box has no imaginable commercial value. Who would pay for a message sent to nobody in particular?” *David Sarnoff*

minimises  $E\{\tilde{y}_{k/k}\tilde{y}_{k/k}^T\}$ .

**Proof:** It follows from (94) that

$$\begin{aligned} E\{\tilde{y}_{k/k}\tilde{y}_{k/k}^T\} &= (C_{1,k} - L_k C_{2,k})P_{k/k-1}(C_{1,k}^T - C_{2,k}^T L_k^T) \\ &\quad + [D_{1,k} \quad -L_k] \begin{bmatrix} Q_k & Q_k D_{2,k}^T \\ D_{2,k} Q_k^T & D_{2,k} Q_k D_{2,k}^T + R_k \end{bmatrix} \begin{bmatrix} D_{1,k}^T \\ -L_k^T \end{bmatrix} \\ &= C_{1,k} P_{k/k-1} C_{1,k}^T - C_{2,k} L_k \Omega_k L_k^T C_{2,k}^T, \end{aligned} \quad (96)$$

which can be expanded to give

$$\begin{aligned} E\{\tilde{y}_{k/k}\tilde{y}_{k/k}^T\} &= C_{1,k} P_{k/k-1} C_{1,k}^T + D_{2,k} Q_k D_{2,k}^T \\ &\quad - (C_{1,k} P_{k/k-1} C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T) \Omega_k^{-1} (C_{1,k} P_{k/k-1} C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T)^T \\ &\quad + (L_k - (C_{1,k} P_{k/k-1} C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T) \Omega_k^{-1}) \Omega_k \\ &\quad + (L_k - (C_{1,k} P_{k/k-1} C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T) \Omega_k^{-1})^T. \end{aligned} \quad (97)$$

By inspection of (97), the filter gain (95) minimises  $E\{\tilde{y}_{k/k}\tilde{y}_{k/k}^T\}$ .

The filter gain (95) has been generalised to include arbitrary  $C_{1,k}$ ,  $D_{1,k}$ , and  $D_{2,k}$ . For state estimation,  $C_2 = I$  and  $D_2 = 0$ , in which case (95) reverts to the simpler form (26). The problem (84) – (88) can be written compactly in the following generalised regulator framework from control theory [13].

$$\begin{bmatrix} x_{k+1} \\ \tilde{y}_{k/k} \\ z_k \end{bmatrix} = \begin{bmatrix} A_k & B_{1,1,k} & 0 \\ C_{1,1,k} & D_{1,1,k} & D_{1,2,k} \\ C_{2,1,k} & D_{2,1,k} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ v_k \\ w_k \end{bmatrix}, \quad (98)$$

where  $B_{1,1,k} = [0 \quad B_k]$ ,  $C_{1,1,k} = C_{1,k}$ ,  $C_{2,1,k} = C_{2,k}$ ,  $D_{1,1,k} = [0 \quad D_{1,k}]$ ,  $D_{1,2,k} = I$  and  $D_{2,1,k} = [I \quad D_{2,k}]$ . With the above definitions, the minimum-variance solution can be written as

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k-1} + K_k (z_k - C_{2,1,k} \hat{x}_{k/k-1}), \quad (99)$$

$$\hat{y}_{1,k/k} = C_{1,1,k} \hat{x}_{k/k-1} + L_k (z_k - C_{2,1,k} \hat{x}_{k/k-1}), \quad (100)$$

where

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“Video won’t be able to hold on to any market it captures after the first six months. People will soon get tired of staring at a plywood box every night.” *Daryl Francis Zanuck*



$$K_k = \left( A_k P_{k/k-1} C_{2,1,k}^T + B_{1,1,k} \begin{bmatrix} R_k & 0 \\ 0 & Q_k \end{bmatrix} D_{2,1,k}^T \right) \left( C_{2,1,k} P_{k/k-1} C_{2,1,k}^T + D_{2,1,k} \begin{bmatrix} R_k & 0 \\ 0 & Q_k \end{bmatrix} D_{2,1,k}^T \right)^{-1}, \quad (101)$$

$$L_k = \left( C_{1,1,k} P_{k/k-1} C_{2,1,k}^T + D_{1,1,k} \begin{bmatrix} R_k & 0 \\ 0 & Q_k \end{bmatrix} D_{2,1,k}^T \right) \left( C_{2,1,k} P_{k/k-1} C_{2,1,k}^T + D_{2,1,k} \begin{bmatrix} R_k & 0 \\ 0 & Q_k \end{bmatrix} D_{2,1,k}^T \right)^{-1}, \quad (102)$$

in which  $P_{k/k-1}$  is the solution of the Riccati difference equation

$$P_{k+1/k} = A_k P_{k/k-1} A_k^T - K_k (C_{2,1,k} P_{k/k-1} C_{2,1,k}^T + D_{2,1,k} \begin{bmatrix} R_k & 0 \\ 0 & Q_k \end{bmatrix} D_{2,1,k}^T) K_k^T + B_{1,1,k} \begin{bmatrix} R_k & 0 \\ 0 & Q_k \end{bmatrix} B_{1,1,k}^T. \quad (103)$$

The application of the solution (99) – (100) to output estimation, input estimation (or equalisation), state estimation and mixed filtering problems is demonstrated in the example below.

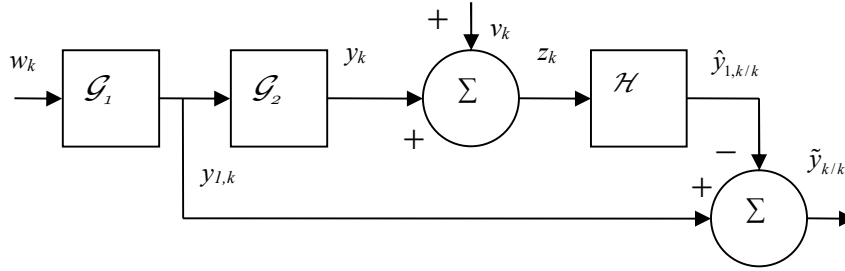


Fig. 8. The mixed filtering and equalisation problem considered in Example 5. The objective is to estimate the output of the plant  $\mathcal{G}_1$  which has been corrupted by the channel  $\mathcal{G}_2$  and the measurement noise  $v_k$ .

**Example 5.**

- (i) For output estimation problems, where  $C_{1,k} = C_{2,k}$  and  $D_{1,k} = D_{2,k}$ , the predictor gain (101) and filter gain (102) are identical to the previously derived (90) and (95), respectively.
- (ii) For state estimation problems, set  $C_{1,k} = I$  and  $D_{1,k} = 0$ .
- (iii) For equalisation problems, set  $C_{1,k} = 0$  and  $D_{1,k} = I$ .
- (iv) Consider a mixed filtering and equalisation problem depicted in Fig. 8, where the output of the plant  $\mathcal{G}_1$  has been corrupted by the channel  $\mathcal{G}_2$ . Assume that  $\mathcal{G}_1$  has the realisation

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“Louis Pasteur’s theory of germs is ridiculous fiction.” *Pierre Pachet*, Professor of Physiology at Toulouse, 1872

$\begin{bmatrix} x_{1,k+1} \\ y_{1,k} \end{bmatrix} = \begin{bmatrix} A_{1,k} & B_{1,k} \\ C_{1,k} & D_{1,k} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ w_k \end{bmatrix}$ . Noting the realisation of the cascaded system  $\mathcal{G}_1 \mathcal{G}_2$  (see Problem 7), the minimum-variance solution can be found by setting  $A_k = \begin{bmatrix} A_{2,k} & B_{2,k} C_{1,k} \\ 0 & A_{1,k} \end{bmatrix}$ ,  $B_{1,k} = \begin{bmatrix} 0 & B_{2,k} D_{1,k} \\ 0 & B_{1,k} \end{bmatrix}$ ,  $B_{1,2,k} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $C_{1,1,k} = \begin{bmatrix} C_{1,k} & 0 \end{bmatrix}$ ,  $C_{2,1,k} = \begin{bmatrix} C_{2,k} & D_{2,k} C_{1,k} \end{bmatrix}$ ,  $D_{1,1,k} = \begin{bmatrix} 0 & D_{1,k} \end{bmatrix}$  and  $D_{2,1,k} = \begin{bmatrix} I & D_{2,k} D_{1,k} \end{bmatrix}$ .

## 4.18 Hybrid Continuous-Discrete Filtering

Often a system's dynamics evolve continuously but measurements can only be observed in discrete time increments. This problem is modelled in [20] as

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (104)$$

$$z_k = C_k x_k + v_k, \quad (105)$$

where  $E\{w(t)\} = 0$ ,  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t - \tau)$ ,  $E\{v_k\} = 0$ ,  $E\{v_k v_k^T\} = R_k \delta_{jk}$  and  $x_k = x(kT_s)$ , in which  $T_s$  is the sampling interval. Following the approach of [20], state estimates can be obtained from a hybrid of continuous-time and discrete-time filtering equations. The predicted states and error covariances are obtained from

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t), \quad (106)$$

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + B(t)Q(t)B^T(t). \quad (107)$$

Define  $\hat{x}_{k/k-1} = \hat{x}(t)$  and  $P_{k/k-1} = P(t)$  at  $t = kT_s$ . The corrected states and error covariances are given by

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k(z_k - C_k \hat{x}_{k/k-1}), \quad (108)$$

$$P_{k/k} = (I - L_k C_k)P_{k/k-1}, \quad (109)$$

where  $L_k = P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}$ . The above filter is a linear system having jumps at the discrete observation times. The states evolve according to the continuous-time dynamics (106) in-between the sampling instants. This filter is applied in [20] for recovery of cardiac dynamics from medical image sequences.

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“Heavier-than-air flying machines are impossible.” *Baron William Thomson Kelvin*

	ASSUMPTIONS	MAIN RESULTS
Signals and system	$E\{w_k\} = E\{v_k\} = 0.$ $E\{w_k w_k^T\} = Q_k$ and $E\{v_k v_k^T\} = R_k$ are known. $A_k, B_k, C_{1,k},$ $C_{2,k}, D_{1,k}, D_{2,k}$ are known.	$x_{k+1} = A_k x_k + B_k w_k$ $x_{k+1} = A_k x_k + D_{2,k} w_k$ $z_k = y_{2,k} + v_k$ $y_{1,k} = C_{1,k} x_k + D_{1,k} w_k$
Filtered state and output		$\hat{x}_{k+1/k} = (A_k - K_k C_{2,k}) \hat{x}_{k/k-1} + K_k z_k$ $\hat{y}_{1,k/k} = (C_{1,k} - L_k C_{2,k}) \hat{x}_{k/k-1} + L_k z_k$
Predictor gain, filter gain and Riccati difference equation	$Q_k > 0, R_k > 0.$	$K_k = (A_k P_{k/k-1} C_{2,k}^T + B_k Q_k D_{2,k}^T)$ $\times (C_{2,k} P_{k/k-1} C_{2,k}^T + D_{2,k} Q_k D_{2,k}^T + R_k)^{-1}$ $L_k = (C_{1,k} P_{k/k-1} C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T)$ $\times (C_{2,k} P_{k/k-1} C_{2,k}^T + D_{2,k} Q_k D_{2,k}^T + R_k)^{-1}$ $P_{k+1/k} = A_k P_{k/k-1} A_k^T + B_k Q_k B_k^T$ $- K_k (C_{2,k} P_{k/k-1} C_{2,k}^T + D_{2,k} Q_k D_{2,k}^T + R_k) K_k^T$

Table 1.1. Main results for the general filtering problem.

## 4.19 Chapter Summary

A linear, time-varying system  $\mathcal{G}_2$  is assumed to have the realisation  $x_{k+1} = A_k x_k + B_k w_k$  and  $y_{2,k} = C_{2,k} x_k + D_{2,k} w_k$ . In the general filtering problem, it is desired to estimate the output of a second reference system  $\mathcal{G}_1$  which is modelled as  $y_{1,k} = C_{1,k} x_k + D_{1,k} w_k$ . The Kalman filter which estimates  $y_{1,k}$  from the measurements  $z_k = y_{2,k} + v_k$  at time  $k$  is listed in Table 1.

If the state-space parameters are known exactly then this filter minimises the predicted and corrected error covariances  $E\{(x_k - \hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1})^T\}$  and  $E\{(x_k - \hat{x}_{k/k})(x_k - \hat{x}_{k/k})^T\}$ , respectively. When there are gaps in the data record, or the data is irregularly spaced, state predictions can be calculated an arbitrary number of steps ahead, at the cost of increased mean-square-error.

The filtering solution is specialised to output estimation with  $C_{1,k} = C_{2,k}$  and  $D_{1,k} = D_{2,k}$ .

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“He was a multimillionaire. Wanna know how he made all of his money? He designed the little diagrams that tell which way to put batteries on.” *Stephen Wright*

In the case of input estimation (or equalisation),  $C_{1,k} = 0$  and  $D_{1,k} = I$ , which results in  $\hat{w}_{k/k} = -L_k C_{2,k} \hat{x}_{k/k-1} + L_k z_k$ , where the filter gain is instead calculated as  $L_k = Q_k D_{2,k}^T (C_{2,k} P_{k/k-1} C_{2,k}^T + D_{2,k} Q_k D_{2,k}^T + R_k)^{-1}$ .

For problems where  $C_{1,k} = I$  (state estimation) and  $D_{1,k} = D_{2,k} = 0$ , the filtered state calculation simplifies to  $\hat{x}_{k/k} = (I - L_k C_{2,k}) \hat{x}_{k/k-1} + L_k z_k$ , where  $\hat{x}_{k/k-1} = A_k \hat{x}_{k-1/k-1}$  and  $L_k = P_{k/k-1} C_{2,k}^T (C_{2,k} P_{k/k-1} C_{2,k}^T + R_k)^{-1}$ . This predictor-corrector form is used to obtain robust, hybrid and extended Kalman filters. When the predicted states are not explicitly required, the state corrections can be calculated from the one-line recursion  $\hat{x}_{k/k} = (I - L_k C_{2,k}) A_{k-1} \hat{x}_{k-1/k-1} + L_k z_k$ .

If the simplifications  $B_k = D_{2,k} = 0$  are assumed and the pair  $(A_k, C_{2,k})$  is retained, the Kalman filter degenerates to the RLS algorithm. However, the cost of this model simplification is an increase in mean-square-error.

## 4.20 Problems

**Problem 1.** Suppose that  $E \left\{ \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} \right\} = \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix}$  and  $E \left\{ \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} \begin{bmatrix} \alpha_k^T & \beta_k^T \end{bmatrix} \right\} = \begin{bmatrix} \Sigma_{\alpha_k \alpha_k} & \Sigma_{\alpha_k \beta_k} \\ \Sigma_{\beta_k \alpha_k} & \Sigma_{\beta_k \beta_k} \end{bmatrix}$ . Show that an estimate of  $\alpha_k$  given  $\beta_k$ , which minimises  $E \{ (\alpha_k - E\{\alpha_k | \beta_k\})(\alpha_k - E\{\alpha_k | \beta_k\})^T \}$ , is given by  $E\{\alpha_k | \beta_k\} = \bar{\alpha} + \Sigma_{\alpha_k \beta_k} \Sigma_{\beta_k \beta_k}^{-1} (\beta_k - \bar{\beta})$ .

**Problem 2.** Derive the predicted error covariance

$P_{k+1/k} = A_k P_{k/k-1} A_k^T - A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1} A_k^T + B_k Q_k B_k^T$  from the state prediction  $\hat{x}_{k+1/k} = A_k \hat{x}_{k/k-1} + K_k (z_k - C_k \hat{x}_{k/k-1})$ , the model  $x_{k+1} = A_k x_k + B_k w_k$ ,  $y_k = C_k x_k$  and the measurements  $z_k = v_k + v_k$ .

**Problem 3.** Assuming the state correction  $\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1})$ , show that the corrected error covariance is given by  $P_{k/k} = P_{k/k-1} - L_k (C_k P_{k/k-1} C_k^T + R_k) L_k^T$ .

**Problem 4** [11], [14], [17], [18], [19]. Consider the standard filter equations

$$\hat{x}_{k/k-1} = A_k \hat{x}_{k-1/k-1},$$

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1}),$$

“But what is it good for?” Engineer at the Advanced Computing Systems Division of IBM, commenting on the micro chip, 1968

$$P_{k/k-1} = A_k P_{k-1/k-1} A_k^T + B_k Q_k B_k^T,$$

$$P_{k/k} = P_{k/k-1} - L_k (C_k P_{k/k-1} C_k^T + R_k) L_k^T,$$

derive the continuous-time filter equations, namely

$$\dot{\hat{x}}(t_k) = A(t_k) \hat{x}(t_k) + K(t_k) (z(t_k) - C(t_k) \hat{x}(t_k)),$$

$$\dot{P}(t_k) = A(t_k) P(t_k) + P(t_k) A^T(t_k) - P(t_k) C^T(t_k) R^{-1}(t_k) C(t_k) P(t_k) + B(t_k) Q(t_k) B^T(t_k),$$

where  $K(t_k) = P(t_k) C(t_k) R^{-1}(t_k)$ . (Hint: Introduce the quantities  $A_k = (I + A(t_k)) \Delta t$ ,  $B(t_k) = B_k$ ,  $C(t_k) = C_k$ ,  $P_{k/k} = Q_k / \Delta t$ ,  $R(t_k) = R_k \Delta t$ ,  $\hat{x}(t_k) = \hat{x}_{k/k}$ ,  $P(t_k) =$

$$P_{k/k}, \quad \dot{\hat{x}}(t_k) = \lim_{\Delta t \rightarrow 0} \frac{\hat{x}_{k/k} - \hat{x}_{k-1/k-1}}{\Delta t}, \quad \dot{P}(t_k) = \lim_{\Delta t \rightarrow 0} \frac{P_{k+1/k} - P_{k/k-1}}{\Delta t} \text{ and } \Delta t = t_k - t_{k-1}.)$$

**Problem 5.** Derive the two-step-ahead predicted error covariance  $P_{k+2/k} = A_{k+1} P_{k+1/k} A_{k+1}^T + B_{k+1} Q_{k+1} B_{k+1}^T$ .

**Problem 6.** Verify that the Riccati difference equation  $P_{k+1/k} = A_k P_{k/k-1} A_k^T - K_k (C_k P_{k/k-1} C_k^T + R_k) K_k^T + B_k Q_k B_k^T$ , where  $K_k = (A_k P_{k/k-1} C_k^T + B_k S_k) (C_k P_{k/k-1} C_k^T + R_k)^{-1}$ , is equivalent to  $P_{k+1/k} = (A_k - K_k C_k) P_{k/k-1} (A_k - K_k C_k)^T + K_k R_k K_k^T + B_k Q_k B_k^T - B_k S_k K_k^T - K_k S_k B_k^T$ .

**Problem 7 [16].** Suppose that the systems  $y_{1,k} = \mathcal{G}_1 w_k$  and  $y_{2,k} = \mathcal{G}_2 w_k$  have the state-space realisations

$$\begin{bmatrix} x_{1,k+1} \\ y_{1,k} \end{bmatrix} = \begin{bmatrix} A_{1,k} & B_{1,k} \\ C_{1,k} & D_{1,k} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ w_k \end{bmatrix} \text{ and } \begin{bmatrix} x_{2,k+1} \\ y_{2,k} \end{bmatrix} = \begin{bmatrix} A_{2,k} & B_{2,k} \\ C_{2,k} & D_{2,k} \end{bmatrix} \begin{bmatrix} x_{2,k} \\ w_k \end{bmatrix}.$$

Show that the system  $y_{3,k} = \mathcal{G}_2 \mathcal{G}_1 w_k$  is given by

$$\begin{bmatrix} x_{1,k+1} \\ y_{3,k} \end{bmatrix} = \begin{bmatrix} A_{1,k} & 0 & B_{1,k} \\ B_{2,k} C_{1,k} & A_{2,k} & B_{2,k} D_{1,k} \\ D_{2,k} C_{1,k} & C_{2,k} & D_{2,k} D_{1,k} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ w_k \end{bmatrix}.$$

## 4.21 Glossary

In addition to the notation listed in Section 2.6, the following nomenclature has been used herein.

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“What sir, would you make a ship sail against the wind and currents by lighting a bonfire under her deck? I pray you excuse me. I have no time to listen to such nonsense.” *Napoléon Bonaparte*

$\mathcal{G}$	A system that is assumed to have the realisation $x_{k+1} = A_k x_k + B_k w_k$ and $y_k = C_k x_k + D_k w_k$ where $A_k$ , $B_k$ , $C_k$ and $D_k$ are time-varying matrices of appropriate dimension.
$Q_k, R_k$	Time-varying covariance matrices of stochastic signals $w_k$ and $v_k$ , respectively.
$\mathcal{G}^H$	Adjoint of $\mathcal{G}$ . The adjoint of a system having the state-space parameters $\{A_k, B_k, C_k, D_k\}$ is a system parameterised by $\{A_k^T, -C_k^T, -B_k^T, D_k^T\}$ .
$\hat{x}_{k/k}$	Filtered estimate of the state $x_k$ given measurements at time $k$ .
$\tilde{x}_{k/k}$	Filtered state estimation error which is defined by $\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}$ .
$P_{k/k}$	Corrected error covariance matrix at time $k$ given measurements at time $k$ .
$L_k$	Time-varying filter gain matrix.
$\hat{x}_{k+1/k}$	Predicted estimate of the state $x_{k+1}$ given measurements at time $k$ .
$\tilde{x}_{k+1/k}$	Predicted state estimation error which is defined by $\tilde{x}_{k+1/k} = x_{k+1} - \hat{x}_{k+1/k}$ .
$P_{k+1/k}$	Predicted error covariance matrix at time $k + 1$ given measurements at time $k$ .
$K_k$	Time-varying predictor gain matrix.
RLS	Recursive Least Squares.

## 4.22 References

- [1] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems", *Transactions of the ASME, Series D, Journal of Basic Engineering*, vol 82, pp. 35 – 45, 1960.
- [2] R. E. Kalman, "New Methods in Wiener Filtering Theory", *Proc. First Symposium on Engineering Applications of Random Function Theory and Probability*, Wiley, New York, pp. 270 – 388, 1963.
- [3] T. Söderström, *Discrete-time Stochastic Systems: Estimation and Control*, Springer-Verlag London Ltd., 2002.
- [4] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1979.
- [5] G. S. Maddala, *Introduction to Econometrics*, Second Edition, Macmillan Publishing Co., New York, 1992.
- [6] C. K. Chui and G. Chen, *Kalman Filtering with Real-Time Applications*, 3<sup>rd</sup> Ed., Springer-Verlag, Berlin, 1999.

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"The horse is here today, but the automobile is only a novelty - a fad." President of *Michigan Savings Bank*

- [7] I. Yaesh and U. Shaked, “ $H_\infty$ -Optimal Estimation – The Discrete Time Case”, *Proceedings of the MTNS*, pp. 261 – 267, Jun. 1991.
- [8] U. Shaked and Y. Theodor, “ $H_\infty$  Optimal Estimation: A Tutorial”, *Proceedings 31<sup>st</sup> IEEE Conference on Decision and Control*, pp. 2278 – 2286, Tucson, Arizona, Dec. 1992.
- [9] F. L. Lewis, L. Xie and D. Popa, *Optimal and Robust Estimation: With an Introduction to Stochastic Control Theory*, Second Edition, Series in Automation and Control Engineering, Taylor & Francis Group, LLC, 2008.
- [10] T. Kailath, A. H. Sayed and B. Hassibi, *Linear Estimation*, Prentice-Hall, Inc., Upper Saddle River, New Jersey, 2000.
- [11] D. Simon, *Optimal State Estimation, Kalman  $H_\infty$  and Nonlinear Approaches*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.
- [12] P. J. Brockwell and R. A. Davis, *Time Series: Theory and Methods*, Second Edition, Springer-Verlag New York, Inc., 1991.
- [13] D. J. N. Limebeer, M. Green and D. Walker, “Discrete-time  $H_\infty$  Control”, *Proceedings 28<sup>th</sup> IEEE Conference on Decision and Control*, Tampa, pp. 392 – 396, Dec., 1989.
- [14] R. G. Brown and P. Y. C. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, Second Edition, John Wiley & Sons, Inc., New York, 1992.
- [15] K. Ogata, *Discrete-time Control Systems*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1987.
- [16] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice-Hall Inc. Englewood Cliffs, New Jersey, 1995.
- [17] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, Inc., New York, 1970.
- [18] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York, 1971.
- [19] A. Gelb, *Applied Optimal Estimation*, The Analytic Sciences Corporation, USA, 197
- [20] S. Tong and P. Shi, “Sampled-Data Filtering Framework for Cardiac Motion Recovery: Optimal Estimation of Continuous Dynamics From Discrete Measurements”, *IEEE Transactions on Biomedical Engineering*, vol. 54, no. 10, pp. 1750 – 1761, Oct. 2007.

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“Airplanes are interesting toys but of no military value.” *Marechal Ferdinand Foch*

## 5. Discrete-Time, Steady-State, Minimum-Variance Filtering

### 5.1 Introduction

This chapter presents the minimum-variance filtering results simplified for the case when the model parameters are time-invariant and the noise processes are stationary. The filtering objective remains the same, namely, the task is to estimate a signal in such a way as to minimise the filter error covariance. A somewhat naïve approach is to apply the standard filter recursions using the time-invariant problem parameters. Although this approach is valid, it involves recalculating the Riccati difference equation solution and filter gain at each time-step, which is computationally expensive. A lower implementation cost can be realised by recognising that the Riccati difference equation solution asymptotically approaches the solution of an algebraic Riccati equation. In this case, the algebraic Riccati equation solution and hence the filter gain can be calculated before running the filter.

The steady-state discrete-time Kalman filtering literature is vast and some of the more accessible accounts [1] – [14] are canvassed here. The filtering problem and the application of the standard time-varying filter recursions are described in Section 5.2. An important criterion for checking whether the states can be uniquely reconstructed from the measurements is observability. For example, sometimes states may be internal or sensor measurements might not be available, which can result in the system having hidden modes. Section 5.3 describes two common tests for observability, namely, checking that an observability matrix or an observability gramian are of full rank. The subject of Riccati equation monotonicity and convergence has been studied extensively by Chan [4], De Souza [5], [6], Bitmead [7], [8], Wimmer [9] and Wonham [10], which is discussed in Section 5.4. Chan, *et al* [4] also showed that if the underlying system is stable and observable then the minimum-variance filter is stable. Section 6 describes a discrete-time version of the Kalman-Yakubovich-Popov Lemma, which states for time-invariant systems that solving a Riccati equation is

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“Science is nothing but trained and organized common sense differing from the latter only as a veteran may differ from a raw recruit: and its methods differ from those of common sense only as far as the guardsman's cut and thrust differ from the manner in which a savage wields his club.” *Thomas Henry Huxley*



equivalent to spectral factorisation. In this case, the Wiener and Kalman filters are the same.

Since the optimal filter is model-based, any unknown model parameters need to be estimated (as explained in Chapter 7) prior to implementation. The estimated parameters can be inexact which leads to degraded filter performance. An iterative frequency weighting procedure is described in Section 5.5 for mitigating the performance degradation.

## 5.2 Time-Invariant Filtering Problem

### 5.2.1 The Time-Invariant Signal Model

A discrete-time time-invariant system (or plant)  $\mathcal{G}: \mathbb{R}^{m \times N} \rightarrow \mathbb{R}^{p \times N}$  is assumed to have the state-space representation

$$x_{k+1} = Ax_k + Bw_k, \quad (1)$$

$$y_k = Cx_k + Dw_k, \quad (2)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$ ,  $w_k$  is a stationary process with  $E\{w_k\} = 0$  and  $E\{w_j w_k^T\} = Q$ . For convenience, the simplification  $D = 0$  is initially assumed within the developments. The nonzero feedthrough matrix,  $D$ , can be accommodated as described in Chapter 4. Observations  $z_k$  of the system output  $y_k$  are again modelled as

$$z_k = y_k + v_k, \quad (3)$$

where  $v_k$  is a stationary measurement noise sequence over an interval  $k \in [1, N]$ , with  $E\{v_k\} = 0$ ,  $E\{w_j v_k^T\} = 0$  and  $E\{v_j v_k^T\} = R$ . The objective is to design a filter  $\mathcal{H}$  that operates on the above measurements and produces an estimate,  $\hat{y}_{k/k} = C_k \hat{x}_{k/k}$ , of  $y_k$  so that the covariance,  $E\{\tilde{y}_{k/k} \tilde{y}_{k/k}^T\}$ , of the filter error,  $\tilde{y}_{k/k} = y_k - \hat{y}_{k/k}$ , is minimised.

### 5.2.2 Application of the Time-Varying Filter Recursions

A naïve but entirely valid approach to state estimation is to apply the standard minimum-variance filter recursions of Chapter 4 for the problem (1) – (3). The predicted and corrected state estimates are given by

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“What happens depends on our way of observing it or the fact that we observe it.” *Werner Heisenberg*

$$\hat{x}_{k+1/k} = (A - K_k C_k) \hat{x}_{k/k-1} + K_k z_k, \quad (4)$$

$$\hat{x}_{k/k} = (I - L_k C_k) \hat{x}_{k/k-1} + L_k z_k, \quad (5)$$

where  $L_k = P_{k/k-1} C^T (C P_{k/k-1} C + R)^{-1}$  is the filter gain,  $K_k = A P_{k/k-1} C^T (C P_{k/k-1} C + R)^{-1}$  is the predictor gain, in which  $P_{k/k-1} = E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\}$  is obtained from the Riccati difference equation

$$P_{k+1} = A P_k A^T - A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T + B Q B^T. \quad (6)$$

As before, the above Riccati equation is iterated forward at each time  $k$  from an initial condition  $P_0$ . A necessary condition for determining whether the states within (1) can be uniquely estimated is observability which is discussed below.

## 5.3 Observability

### 5.3.1 The Discrete-time Observability Matrix

Observability is a fundamental concept in system theory. If a system is unobservable then it will not be possible to recover the states uniquely from the measurements. The pair  $(A, C)$  within the discrete-time system (1) – (2) is defined to be completely observable if the initial states,  $x_0$ , can be uniquely determined from the known inputs  $w_k$  and outputs  $y_k$  over an interval  $k \in [0, N]$ . A test for observability is to check whether an observability matrix is of full rank. The discrete-time observability matrix, which is defined in the lemma below, is the same the continuous-time version. The proof is analogous to the presentation in Chapter 3.

**Lemma 1 [1], [2]:** *The discrete-time system (1) – (2) is completely observable if the observability matrix*

$$O_N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix}, N \geq n - 1, \quad (7)$$

*is of rank  $n$ .*

**Proof:** *Since the input  $w_k$  is assumed to be known, it suffices to consider the unforced system*

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“You affect the world by what you browse.” *Tim Berners-Lee*

$$x_{k+1} = Ax_k, \quad (8)$$

$$y_k = Cx_k. \quad (9)$$

It follows from (8) – (9) that

$$\begin{aligned} y_0 &= Cx_0 \\ y_1 &= Cx_1 = CAx_0 \\ y_2 &= Cx_2 = CA^2x_0 \\ &\vdots \\ y_N &= Cx_N = CA^Nx_0, \end{aligned} \quad (10)$$

which can be written as

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = C \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x_0. \quad (11)$$

From the Cayley-Hamilton Theorem,  $A^k$ , for  $k \geq n$ , can be expressed as a linear combination of  $A^0, A^1, \dots, A^{n-1}$ . Thus, with  $N \geq n - 1$ , equation (11) uniquely determines  $x_0$  if  $O_N$  has full rank  $n$ .

Thus, if  $O_N$  is of full rank then its inverse exists and so  $x_0$  can be uniquely recovered as  $x_0 = O_N^{-1}y$ . Observability is a property of the deterministic model equations (8) – (9). Conversely, if the observability matrix is not rank  $n$  then the system (1) – (2) is termed unobservable and the unobservable states are called unobservable modes.

### 5.3.2 Discrete-time Observability Gramians

Alternative tests for observability arise by checking the rank of one of the observability gramians that are described below.

**Lemma 2:** The pair  $(A, C)$  is completely observable if the observability gramian

$$W_N = O_N^T O_N = \sum_{k=0}^N (A^T)^k C^T C A^k, \quad N \geq n-1 \quad (12)$$

---

“It is a good morning exercise for a research scientist to discard a pet hypothesis every day before breakfast.” *Konrad Zacharias Lorenz*

is of full rank.

**Proof:** It follows from (8) – (9) that

$$y^T y = x_0^T \begin{bmatrix} I & A^T & (A^T)^2 & \cdots & (A^T)^N \end{bmatrix} C^T C \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x_0. \quad (13)$$

From the Cayley-Hamilton Theorem,  $A^k$ , for  $k \geq n$ , can be expressed as a linear combination of  $A^0, A^1, \dots, A^{n-1}$ . Thus, with  $N = n - 1$ ,

$$y^T y = x_0^T O_N^T O_N x_0 = x_0^T W_N x_0 = x_0^T \left( \sum_{k=0}^{n-1} (A^T)^k C^T C A^k \right) x_0 \quad (14)$$

is unique provided that  $W_N$  is of full rank.

It is shown below that an equivalent observability gramian can be found from the solution of a Lyapunov equation.

**Lemma 3:** Suppose that the system (8) – (9) is stable, that is,  $|\lambda_i(A)| < 1$ ,  $i = 1$  to  $n$ , then the pair  $(A, C)$  is completely observable if the nonnegative symmetric solution of the Lyapunov equation

$$W = A^T W A + C^T C. \quad (15)$$

is of full rank.

**Proof:** Pre-multiplying  $C^T C = W - A^T W A$  by  $(A^T)^k$ , post-multiplying by  $A^k$  and summing from  $k = 0$  to  $N$  results in

$$\begin{aligned} \sum_{k=0}^N (A^T)^k C^T C A^k &= \sum_{k=0}^N (A^T)^k W A^k - \sum_{k=0}^N (A^T)^{k+1} W A^{k+1} \\ &= W_N - (A^T)^{k+1} W_N A^{k+1}. \end{aligned} \quad (16)$$

Since  $\lim_{k \rightarrow \infty} (A^T)^{k+1} W_N A^{k+1} = 0$ , by inspection of (16),  $W = \lim_{k \rightarrow \infty} W_N$  is a solution of the Lyapunov equation (15). Observability follows from Lemma 2.

It is noted below that observability is equivalent to asymptotic stability.

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“We follow abstract assumptions to see where they lead, and then decide whether the detailed differences from the real world matter.” *Clinton Richard Dawkins*

**Lemma 4 [3]:** Under the conditions of Lemma 3,  $x_0 \in \ell_2$  implies  $y \in \ell_2$ .

**Proof:** It follows from (16) that  $\sum_{k=0}^N (A^T)^k C^T C A^k \leq W_N$  and therefore

$$\|y\|_2^2 = \sum_{k=0}^N y_k^T y_k = x_0^T \left( \sum_{k=0}^N (A^T)^k C^T C A^k \right) x_0 \leq x_0^T W_N x_0,$$

from which the claim follows.

Another criterion that is encountered in the context of filtering and smoothing is reachability. A linear time-invariant system is said to be reachable when all its modes are excited. Reachability is discussed in Chapters 8 and 11.

**Example 1.** (i) Consider a stable second-order system with  $A = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.4 \end{bmatrix}$  and  $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ . The observability matrix from (7) and the observability gramian from (12) are  $O_1 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.1 & 0.6 \end{bmatrix}$  and  $W_1 = O_1^T O_1 = \begin{bmatrix} 1.01 & 1.06 \\ 1.06 & 1.36 \end{bmatrix}$ , respectively. It can easily be verified that the solution of the Lyapunov equation (15) is  $W = \begin{bmatrix} 1.01 & 1.06 \\ 1.06 & 1.44 \end{bmatrix} = W_4$  to three significant figures. Since  $\text{rank}(O_1) = \text{rank}(W_1) = \text{rank}(W_4) = 2$ , the pair  $(A, C)$  is observable.

(ii) Now suppose that measurements of the first state are not available, that is,  $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ . Since  $O_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0.4 \end{bmatrix}$  and  $W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1.16 \end{bmatrix}$  are of rank 1, the pair  $(A, C)$  is unobservable. This system is detectable because the unobservable mode is stable.

## 5.4 Riccati Equation Properties

### 5.4.1 Monotonicity

It will be shown below that the solution  $P_{k+1/k}$  of the Riccati difference equation (6) monotonically approaches a steady-state asymptote, in which case the gain is also time-invariant and can be precalculated. Establishing monotonicity requires the following result. It is well known that the difference between the solutions of two Riccati equations also obeys a Riccati equation, see Theorem 4.3 of [4], (2.12)

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“The way to succeed is to double your failure rate.” *Thomas Watson, Founder of IBM*

of [5], Lemma 3.1 of [6], (4.2) of [7], Lemma 10.1 of [8], (2.11) of [9] and (2.4) of [10].

**Theorem 1: Riccati Equation Comparison Theorem [4] – [10]:** Suppose for a  $t \geq 0$  and for all  $k \geq 0$  the two Riccati difference equations

$$P_{t+k} = AP_{t+k-1}A^T - AP_{t+k-1}C^T(CP_{t+k-1}C^T + R)^{-1}CP_{t+k-1}A^T + BQB^T, \quad (17)$$

$$P_{t+k+1} = AP_{t+k}A^T - AP_{t+k}C^T(CP_{t+k}C^T + R)^{-1}CP_{t+k}A^T + BQB^T, \quad (18)$$

have solutions  $P_{t+k} \geq 0$  and  $P_{t+k-1} \geq 0$ , respectively. Then  $\bar{P}_{t+k} = P_{t+k} - P_{t+k+1}$  satisfies

$$\bar{P}_{t+k+1} = \bar{A}_{t+k+1}\bar{P}_{t+k}\bar{A}_{t+k}^T - \bar{A}_{t+k+1}\bar{P}_{t+k}C^T(C\bar{P}_{t+k}C^T + \bar{R}_k)^{-1}C\bar{P}_{t+k}\bar{A}_{t+k+1}^T, \quad (19)$$

where  $\bar{A}_{t+k+1} = A - AP_{t+k-1}C^T(CP_{t+k-1}C^T + \bar{R}_{t+k+1})^{-1}C_{t+k}$  and  $\bar{R}_{t+k} = CP_{t+k}C^T + R$ .

The above result can be verified by substituting  $\bar{A}_{t+k+1}$  and  $\bar{R}_{t+k+1}$  into (19). The above theorem is used below to establish Riccati difference equation monotonicity.

**Theorem 2 [6], [9], [10], [11]:** Under the conditions of Theorem 1, suppose that the solution of the Riccati difference equation (19) has a solution  $\bar{P}_{t+k} \geq 0$  for a  $t \geq 0$  and  $k = 0$ . Then  $P_{t+k} \geq P_{t+k+1}$  for all  $k \geq 0$ .

**Proof:** The assumption  $\bar{P}_{t+k} \geq 0$  is the initial condition for an induction argument. For the induction step, it follows from  $C\bar{P}_{t+k}C^T(C\bar{P}_{t+k}C^T + \bar{R}_k)^{-1} \leq I$  that  $\bar{P}_{t+k} \leq \bar{P}_{t+k}C^T(C\bar{P}_{t+k}C^T + \bar{R}_k)^{-1}C\bar{P}_{t+k}$ , which together with Theorem 1 implies  $\bar{P}_{t+k} \geq 0$ .

The above theorem serves to establish conditions under which a Riccati difference equation solution monotonically approaches its steady state solution. This requires a Riccati equation convergence result which is presented below.

## 5.4.2 Convergence

When the model parameters and noise statistics are constant then the predictor gain is also time-invariant and can be pre-calculated as

$$K = APC^T(CPC^T + R)^{-1}, \quad (20)$$

---

“We know very little, and yet it is astonishing that we know so much, and still more astonishing that so little knowledge can give us so much power”. *Bertrand Arthur William Russell*

where  $P$  is the symmetric positive definite solution of the algebraic Riccati equation

$$P = APA^T - APC^T(CPC^T + R)^{-1}CPA^T + BQB^T \quad (21)$$

$$= (A - KC)P(A - KC)^T + BQB^T + KRK^T. \quad (22)$$

A real symmetric nonnegative definite solution of the Algebraic Riccati equation (21) is said to be a *strong solution* if the eigenvalues of  $(A - KC)$  lie inside or on the unit circle [4], [5]. If there are no eigenvalues on the unit circle then the strong solution is termed the *stabilising solution*. The following lemma by Chan, Goodwin and Sin [4] sets out conditions for the existence of solutions for the algebraic Riccati equation (21).

**Lemma 5 [4]:** *Provided that the pair  $(A, C)$  is detectable, then*

- i) *the strong solution of the algebraic Riccati equation (21) exists and is unique;*
- ii) *if  $A$  has no modes on the unit circle then the strong solution coincides with the stabilising solution.*

A detailed proof is presented in [4]. If the linear time-invariant system (1) – (2) is stable and completely observable and the solution  $P_k$  of the Riccati difference equation (6) is suitably initialised, then in the limit as  $k$  approaches infinity,  $P_k$  will asymptotically converge to the solution of the algebraic Riccati equation. This convergence property is formally restated below.

**Lemma 6 [4]:** *Subject to:*

- i) *the pair  $(A, C)$  is observable;*
- ii)  *$|\lambda_i(A)| \leq 1$ ,  $i = 1$  to  $n$ ;*
- iii)  *$(P_0 - P) \geq 0$ ;*

*then the solution of the Riccati difference equation (6) satisfies*

$$\lim_{k \rightarrow \infty} P_k = P. \quad (23)$$

A proof appears in [4]. This important property is used in [6], which is in turn employed within [7] and [8]. Similar results are reported in [5], [13] and [14]. Convergence can occur exponentially fast which is demonstrated by the following numerical example.

**Example 2.** Consider an output estimation problem where  $A = 0.9$  and  $B = C = Q = R = 1$ . The solution to the algebraic Riccati equation (21) is  $P = 1.4839$ . Some calculated solutions of the Riccati difference equation (6) initialised with  $P_0 = 10P$  are shown in Table 1. The data in the table demonstrate that the Riccati difference equation solution converges to the algebraic Riccati equation solution, which illustrates the Lemma.

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“Great is the power of steady misrepresentation - but the history of science shows how, fortunately, this power does not endure long”. *Charles Robert Darwin*

$k$	$P_k$	$P_{k-1} - P_k$
1	1.7588	13.0801
2	1.5164	0.2425
5	1.4840	$4.7955 \cdot 10^{-4}$
10	1.4839	$1.8698 \cdot 10^{-8}$

Table. 1. Solutions of (21) for Example 2.

## 5.5 The Steady-State Minimum-Variance Filter

### 5.5.1 State Estimation

The formulation of the steady-state Kalman filter (which is also known as the limiting Kalman filter) follows by allowing  $k$  to approach infinity and using the result of Lemma 6. That is, the filter employs fixed gains that are calculated using the solution of the algebraic Riccati equation (21) instead of the Riccati difference equation (6). The filtered state is calculated as

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L(z_k - C\hat{x}_{k/k-1}) = (I - LC)\hat{x}_{k/k-1} + Lz_k, \quad (24)$$

where  $L = PC^T(CPC^T + R)^{-1}$  is the time-invariant filter gain, in which  $P$  is the solution of the algebraic Riccati equation (21). The predicted state is given by

$$\hat{x}_{k+1/k} = A\hat{x}_{k/k} = (A - KC)\hat{x}_{k/k-1} + Kz_k, \quad (25)$$

where the time-invariant predictor gain,  $K$ , is calculated from (20).

### 5.5.2 Asymptotic Stability

The asymptotic stability of the filter (24) – (25) is asserted in two ways. First, recall from Lemma 4 (ii) that if  $|\lambda_i(A)| < 1$ ,  $i = 1$  to  $n$ , and the pair  $(A, C)$  is completely observable, then  $|\lambda_i(A - KC)| < 1$ ,  $i = 1$  to  $n$ . That is, since the eigenvalues of the filter's state matrix are within the unit circle, the filter is asymptotically stable. Second, according to Lyapunov stability theory [1], the

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"The scientists of today think deeply instead of clearly. One must be sane to think clearly, but one can think deeply and be quite insane." *Nikola Tesla*



unforced system (8) is asymptotically stable if there exists a scalar continuous function  $V(x)$ , satisfying the following.

- (i)  $V(x) > 0$  for  $x \neq 0$ .
- (ii)  $V(x_{k+1}) - V(x_k) \leq 0$  for  $x_k \neq 0$ .
- (iii)  $V(0) = 0$ .
- (iv)  $V(x) \rightarrow \infty$  as  $\|x\|_2 \rightarrow \infty$ .

Consider the function  $V(x_k) = x_k^T P x_k$  where  $P$  is a real positive definite symmetric matrix. Observe that  $V(x_{k+1}) - V(x_k) = x_{k+1}^T P x_{k+1} - x_k^T P x_k = x_k^T (A^T P A - P) x_k \leq 0$ . Therefore, the above stability requirements are satisfied if for a real symmetric positive definite  $Q$ , there exists a real symmetric positive definite  $P$  solution to the Lyapunov equation

$$A P A^T - P = -Q. \quad (26)$$

By inspection, the design algebraic Riccati equation (22) is of the form (26) and so the filter is said to be stable in the sense of Lyapunov.

### 5.5.3 Output Estimation

For output estimation problems, the filter gain,  $L$ , is calculated differently. The output estimate is given by

$$\begin{aligned} \hat{y}_{k/k} &= C \hat{x}_{k/k} \\ &= C \hat{x}_{k/k-1} + L(z_k - C \hat{x}_{k/k-1}) \\ &= (C - LC) \hat{x}_{k/k-1} + L z_k, \end{aligned} \quad (27)$$

where the filter gain is now obtained by  $L = C P C^T (C P C^T + R)^{-1}$ . The output estimation filter (24) – (25) can be written compactly as

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \hat{y}_{k/k} \end{bmatrix} = \begin{bmatrix} (A - KC) & K \\ (C - LC) & L \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix}, \quad (28)$$

from which its transfer function is

$$H_{OE}(z) = (C - LC)(zI - A + KC)^{-1} K + L. \quad (29)$$

---

“Any intelligent fool can make things bigger and more complex... It takes a touch of genius - and a lot of courage to move in the opposite direction.” *Albert Einstein*

## 5.6 Equivalence of the Wiener and Kalman Filters

As in continuous-time, solving a discrete-time algebraic Riccati equation is equivalent to spectral factorisation and the corresponding Kalman-Yakubovich-Popov Lemma (or Positive Real Lemma) is set out below. A proof of this Lemma makes use of the following identity

$$P - APA^T = (zI - A)P(z^{-1}I - A^T) + AP(z^{-1}I - A^T) + (zI - A)PA^T. \quad (30)$$

**Lemma 7.** Consider the spectral density matrix

$$\Delta\Delta^H(z) = \begin{bmatrix} C(zI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - A^T)^{-1}C^T \\ I \end{bmatrix}. \quad (31)$$

Then the following statements are equivalent.

- (i)  $\Delta\Delta^H(e^{j\omega}) \geq 0$ , for all  $\omega \in (-\pi, \pi)$ .
- (ii)  $\begin{bmatrix} BQB^T - P + APA^T & APC^T \\ CPA^T & CPC^T + R \end{bmatrix} \geq 0$ .
- (iii) There exists a nonnegative solution  $P$  of the algebraic Riccati equation (21).

**Proof:** Following the approach of [12], to establish equivalence between (i) and (iii), use (21) within (30) to obtain

$$\begin{aligned} BQB^T - APC^T(CPC^T + R)CPA^T &= (zI - A)P(z^{-1}I - A^T) \\ &\quad + AP(z^{-1}I - A^T) + (zI - A)PA^T. \end{aligned} \quad (32)$$

Premultiplying and postmultiplying (32) by  $C(zI - A)^{-1}$  and  $(z^{-1}I - A^T)^{-1}C^T$ , respectively, results in  $C(zI - A)^{-1}(BQB^T - APC^T\Omega CPA^T)(z^{-1}I - A^T)^{-1}C^T = CPC^T + C(zI - A)^{-1}APC^T + CPA^T(z^{-1}I - A^T)^{-1}C^T$ , where  $\Omega = CPC^T + R$ . Hence,

$$\begin{aligned} \Delta\Delta^H(z) &= GQG^H(z) + R = C(zI - A)^{-1}BQB^T(z^{-1}I - A^T)^{-1}C^T + R \\ &= C(zI - A)^{-1}APC^T\Omega CPA^T(z^{-1}I - A^T)^{-1}C^T \\ &\quad + C(zI - A)^{-1}APC^T + CPA^T(z^{-1}I - A^T)^{-1}C^T + \Omega \\ &= (C(zI - A)^{-1}K + I)\Omega(K^T(z^{-1}I - A^T)^{-1}C^T + I) \\ &\geq 0 \end{aligned} \quad (33)$$

---

“The telephone did not come into existence from the persistent improvement of the postcard.” Amit Kalantri

The Schur complement formula can be used to verify the equivalence of (ii) and (iii).

In Chapter 2, it is shown that the transfer function matrix of the optimal Wiener solution for output estimation is given by

$$H_{OE}(z) = I - R\{\Delta^{-H}\}_+ \Delta^{-1}(z), \quad (34)$$

where  $\{\}_+$  denotes the causal part. This filter produces estimates  $\hat{y}_{k/k}$  from measurements  $z_k$ . By inspection of (33) it follows that the spectral factor is

$$\Delta(z) = C(zI - A)^{-1} K \Omega^{1/2} + \Omega^{1/2}. \quad (35)$$

The Wiener output estimator (34) involves  $\Delta^{-1}(z)$  which can be found using (35) and a special case of the matrix inversion lemma, namely,  $[I + C(zI - A)^{-1}K]^{-1} = I - C(zI - A + KC)^{-1}K$ . Thus, the spectral factor inverse is

$$\Delta^{-1}(z) = \Omega^{-1/2} - \Omega^{-1/2}C(zI - A + KC)^{-1}K. \quad (36)$$

It can be seen from (36) that  $\{\Delta^{-H}\}_+ = \Omega^{-1/2}$ . Recognising that  $I - R\Omega^{-1} = (CPC^T + R)(CPC^T + R)^{-1} - R(CPC^T + R)^{-1} = CPC^T(CPC^T + R)^{-1} = L$ , the Wiener filter (34) can be written equivalently

$$\begin{aligned} H_{OE}(z) &= I - R\Omega^{-1}\Delta^{-1}(z) \\ &= I - R\Omega^{-1} + R\Omega^{-1}C(zI - A + KC)^{-1}K \\ &= L + (C - LC)(zI - A + KC)^{-1}K, \end{aligned} \quad (37)$$

which is identical to the transfer function matrix of the Kalman filter for output estimation (29). In Chapter 2, it is shown that the transfer function matrix of the input estimator (or equaliser) for proper, stable, minimum-phase plants is

$$H_{IE}(z) = G^{-1}(z)(I - R\{\Delta^{-H}\}_+ \Delta^{-1}(z)). \quad (38)$$

Substituting (35) into (38) gives

$$H_{IE}(z) = G^{-1}(z)H_{OE}(z). \quad (39)$$

The above Wiener equaliser transfer function matrices require common poles and zeros to be cancelled. Although the solution (39) is not minimum-order (since some pole-zero cancellations can be made), its structure is instructive. In particular, an estimate of  $w_k$  can be obtained by operating the plant inverse on

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“It is not the possession of truth, but the success which attends the seeking after it, that enriches the seeker and brings happiness to him.” *Max Karl Ernst Ludwig Planck*

$\hat{y}_{k/k}$ , provided the inverse exists. It follows immediately from  $L = CPC^T (CPC^T + R)^{-1}$  that

$$\lim_{R \rightarrow 0} L = I. \quad (40)$$

By inspection of (34) and (40), it follows that

$$\lim_{R \rightarrow 0} \sup_{\omega \in [-\pi, \pi]} |H_{OE}(e^{j\omega})| = I. \quad (41)$$

Thus, under conditions of diminishing measurement noise, the output estimator will be devoid of dynamics and its maximum magnitude will approach the identity matrix. Therefore, for proper, stable, minimum-phase plants, the equaliser asymptotically approaches the plant inverse as the measurement noise becomes negligible, that is,

$$\lim_{R \rightarrow 0} H_{IE}(z) = G^{-1}(z). \quad (42)$$

Time-invariant output and input estimation are demonstrated below.

**Example 3.** Consider a time-invariant input estimation problem in which the plant is given by

$$\begin{aligned} G(z) &= (z + 0.9)^2(z + 0.1)^{-2} \\ &= (z^2 + 1.8z + 0.81)(z^2 + 0.2z + 0.01)^{-1} \\ &= (1.6z + 0.8)(z^2 + 0.2z + 0.01)^{-1} + 1, \end{aligned}$$

together with  $Q = 1$  and  $R = 0.0001$ . The controllable canonical form (see Chapter

1) yields the parameters  $A = \begin{bmatrix} -0.2 & -0.1 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $C = [1.6 \quad 1.8]$  and  $D = 1$ .

From Chapter 4, the corresponding algebraic Riccati equation is  $P = APA^T - K\Omega K^T + BQB^T$ , where  $K = (APC^T + BQD^T)\Omega^{-1}$  and  $\Omega = CPC^T + R + DQD^T$ . The minimum-variance output estimator is calculated as

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \hat{y}_{k/k} \end{bmatrix} = \begin{bmatrix} (A - KC) & K \\ (C - LC) & L \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix},$$

where  $L = (CPC^T + DQD^T)\Omega^{-1}$ . The solution  $P = \begin{bmatrix} 0.0026 & -0.0026 \\ -0.0026 & 0.0026 \end{bmatrix}$  for the algebraic Riccati equation was found using the Hamiltonian solver within *Matlab*<sup>®</sup>.

---

“There is no result in nature without a cause; understand the cause and you will have no need of the experiment.” *Leonardo di ser Piero da Vinci*

```

w=sqrt(Q)*randn(N,1);           % process noise
x=[0;0];                         % initial state
for k = 1:N
    y(k) = C*x + D*w(k);         % plant output
    x = A*x + B*w(k);
end
v=sqrt(R)*randn(1,N);           % measurement noise
z = y + v;                       % measurement
omega=C*P*(C') + D*Q*(D') + R;
K = (A*P*(C') + B*Q*(D'))*inv(omega); % predictor gain
L = Q*(D')*inv(omega);           % equaliser gain
x=[0;0];                         % initial state
for k = 1:N
    w_estimate(k) = - L*C*x + L*z(k); % equaliser output
    x = (A - K*C)*x + K*z(k);       % predicted state
end

```

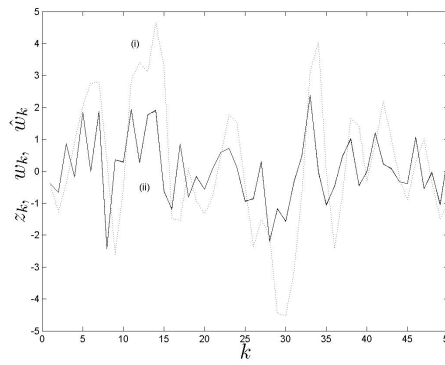
Fig. 1. Fragment of *Matlab*<sup>®</sup> script for Example 3.

Fig. 2. Sample trajectories for Example 5: (i) measurement sequence (dotted line); (ii) actual and estimated process noise sequences (superimposed solid lines).

The resulting transfer function of the output estimator is

$$H_{OE}(z) = (z + 0.9)^2(z + 0.9)^{-2},$$

which illustrates the low-measurement noise asymptote (41). The minimum-variance input estimator is calculated as

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \hat{w}_{k/k} \end{bmatrix} = \begin{bmatrix} (A-KC) & K \\ -LC & L \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix},$$

where  $L = QD^TQ^{-1}$ . The input estimator transfer function is

---

“Your theory is crazy, but it’s not crazy enough to be true.” *Niels Henrik David Bohr*

$$H_{IE}(z) = (z + 0.1)^2 (z + 0.9)^{-2},$$

which corresponds to the inverse of the plant and illustrates the asymptote (42). A simulation was generated based on the fragment of *Matlab*<sup>®</sup> script shown in Fig. 1 and some sample trajectories are provided in Fig. 2. It can be seen from the figure that the actual and estimated process noise sequences are superimposed, which demonstrates that an equaliser can be successful when the plant is invertible and the measurement noise is sufficiently low. In general, when measurement noise is not insignificant, the asymptotes (41) – (42) will not apply, as the minimum-variance equaliser solution will involve a trade-off between inverting the plant and filtering the noise.

## 5.7 Frequency Weighted Filtering

### 5.7.1 Overview

Frequency weighting can be used in filter and controller designs to manage performance within bands of interest. For example, Grimble *et al.* [15] employ a stable, minimum-phase frequency shaping function to weight the average power of the error spectral density in the design of least squares and  $H_\infty$  polynomial filters. Zang *et al.* defined frequency weighting functions as ratios of prediction error and exogenous signal spectra in an iterative controller design [16]. Pots *et al.* chose a frequency weighting function to attenuate noise in a specified frequency band within a robust controller [17]. Perceptual weighting functions based on ratios of linear prediction coefficients can be used within speech codecs [18].

It is shown below that frequency weighting can be applied to improve on the performance of filters which rely on inexact modelling assumptions. Parameter uncertainty can be also accommodated explicitly using robust designs as described in Chapter 9.

### 5.7.2 Problem Definition

Consider a system  $\mathcal{G}$  having the realisation (1) – (2) with measurements (3), where  $y_k \in \mathbb{R}$ ,  $C \in \mathbb{R}^{1 \times n}$  and  $D = 0$ . A filter solution  $\mathcal{H}$  is desired that produces estimates  $\hat{y}_k$  of  $y_k$  from the observations so that the energy of the estimation error

$$\tilde{y}_k = \mathcal{W}(y - \hat{y}) \tag{43}$$

is minimised, where  $\mathcal{W}: \mathbb{R} \rightarrow \mathbb{R}$  is a causal frequency-weighting system that specifies a band of interest.

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“Clear thinking requires courage rather than intelligence.” *Thomas Stephen Szasz*

The estimation problem is depicted in Fig 3. It can be seen from the figure that the error is generated by  $\tilde{y}_k = \mathcal{R} \begin{bmatrix} w \\ v \end{bmatrix}$ , where  $\mathcal{R} = \mathcal{W}[(\mathcal{G} - \mathcal{H}\mathcal{G}) \quad -\mathcal{H}]$ , see [21]. Let  $(.)^H$  and  $(.)^{-H}$  denote the Hermitian conjugate and its inverse. Define  $\Phi_{\tilde{y}\tilde{y}} = \mathcal{R} \begin{bmatrix} \sigma_w^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \mathcal{R}^H$  to be the power spectral density of  $\tilde{y}_k$ . The performance objective is to find a solution  $\mathcal{H}$  that minimises  $\|\Phi_{\tilde{y}\tilde{y}}\|_2$ , where  $\|\cdot\|_2$  denotes the 2-norm.

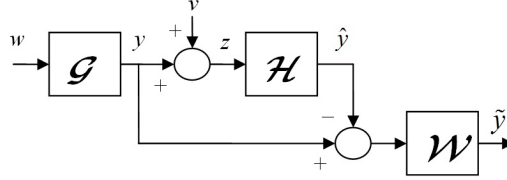


Fig. 3. The frequency-weighted estimation problem. The objective is to design a filter  $\mathcal{H}$  that produces estimates  $\hat{y}$  that minimise the energy of the frequency-weighted estimation error  $\tilde{y}$ .

Usually, in the absence of frequency weighting, *i.e.*,  $\mathcal{W} = 1$ , the optimal filter minimises  $\|\Phi_{\tilde{y}\tilde{y}}\|_2$ , provided that the modelling assumptions are correct. In practice, filters and smoothers are implemented using estimated parameters which can result in degraded performance. Parameter estimation algorithms have been previously investigated in [22] – [23] and it was found that the estimates of  $A$  and  $\sigma_w^2$  only approach the actual values when the measurement noise is negligible. Here, the attention is directed to problems where estimates of  $A$ ,  $\sigma_w^2$  are inexact and significant measurement noise of known variance is present. The problem of interest is to investigate whether a suitable  $\mathcal{W}$  can be designed to improve filter performance.

### 5.7.3 Optimal Filter Solution

Define  $\Delta\Delta^H = \mathcal{G}\mathcal{G}^H\sigma_w^2 + \sigma_v^2$ , in which the spectral factor  $\Delta: \mathbb{R} \rightarrow \mathbb{R}$  is causal, namely,  $\Delta$  and its inverse,  $\Delta^{-1}$ , are bounded systems that proceed forward in time. Following the approach of [21], the optimal frequency-weighted smoother and filter are developed below.

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“Success is a lousy teacher. It seduces smart people into thinking they can’t lose.” *Bill Gates*

**Theorem 1:** For the above estimation problem, the filter solution

$$\mathcal{H} = \mathcal{W}^{-1} \{ \mathcal{G} \mathcal{G}^H \Delta^{-H} \}_+ \Delta^{-1} \sigma_w^2 \quad (44)$$

minimises  $\| \{ \tilde{y} \tilde{y}^H \}_+ \|_2 = \| \{ \Phi_{\tilde{y}\tilde{y}}^{(1)} \}_+ \|_2$ , where  $\{ \cdot \}_+$  denotes the causal part.

The filter (44) can be written as

$$\mathcal{H} = \mathcal{W}^{-1} (I - \sigma_v^2 \{ \Delta^{-H} \}_+ \Delta^{-1}) . \quad (45)$$

### 5.7.4 Unweighted Filter Realisation

A realisation for the filter (45) is set out below. The unweighted filtered output estimates  $\hat{y}^{(1)} \in \mathbb{R}$  are realised by

$$\hat{x}_{k+1} = (A - KC) \hat{x}_k + K z_k, \quad \hat{x}_0 = 0, \quad (46)$$

$$\hat{y}_k^{(1)} = (C - LC) \hat{x}_k + L z_k, \quad (47)$$

where  $L = PC\Omega^{-1}$  is the filter gain,  $K = APC^T\Omega^{-1}$  is the predictor gain,  $\Omega = CPC^T + \sigma_v^2$ , in which  $P = P^T > 0$  is the solution of the algebraic Riccati equation  $P = APA^T - K\Omega K^T + BB^T\sigma_w^2$ .

Denote the sequence of unweighted output estimates from (47) by  $\hat{y}^{(1)} = [\hat{y}_1^{(1)} \dots, \hat{y}_N^{(1)}]^T$ . It can be seen from (44) - (45) that frequency weighting can be applied independently of the filter designs and increases the order of the solutions. In the interest of minimising calculation cost, procedures for designing a *first-order* system,  $\mathcal{W}^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ , which produces frequency-weighted output estimates from  $\hat{y}^{(1)}$  are described in the next section.

### 5.7.5 Frequency Weighting Method

Suppose that the filter (46) - (47) designed with estimated  $A$  and  $\sigma_w^2$  has produced  $\hat{y}^{(1)}$ . Let

$$\tilde{y}^{(j)} = z - \hat{y}^{(j)}, \quad (48)$$

---

“Wall street people learn nothing and forget everything.” *Benjamin Graham*



$j = 1$ , denote the output estimation error exhibited by an unweighted filter. The approach taken herein is to assume that a moving-average-order-1 (MA1) system generates  $\tilde{y}^{(j)}$  which can be identified and used to design a frequency weighting. In particular, assume that

$$\tilde{y}^{(j)} = \mathcal{W}_j^{-1} u, \quad (49)$$

where  $u$  is a hypothetical stochastic input sequence and  $\mathcal{W}_j$  is an identified frequency weighting system. From the forms (44) and (45), an improved frequency-weighted estimate is obtained as

$$\hat{y}^{(j+1)} = \mathcal{W}_j^{-1} \hat{y}^{(j)}. \quad (50)$$

It will be demonstrated in a subsequent example that the above approach may be repeated with  $j = M$  iterations in (48) – (50). The combined  $\mathcal{W}$  is given by  $\mathcal{W} = \mathcal{W}_M \cdots \mathcal{W}_2 \mathcal{W}_1 \mathcal{W}_0$ .

### 5.7.6 Frequency Weighting Design

In respect of assumption (49), suppose that  $\tilde{y}_k^{(j)}$  can be generated by an MA1 system

$$\tilde{y}_k^{(j)} = u_k + b_j u_{k-1}, \quad (51)$$

where  $b_j, u_k \in \mathbb{R}$ . The unknown  $b_j$  and  $u_k$  may be estimated from  $\tilde{y}_k^{(j)}$  by minimising an objective function  $F$  within the following procedure.

**Procedure 1 [25]:** In respect of the MA1 system (51), assume that an initial estimate  $\hat{b}_j^{(1)}$  of  $b_j$  is available and let  $\hat{u}_1^{(1)} = \tilde{y}_1$ . Subsequent estimates  $(b_j - \hat{b}_j^{(i+1)} \hat{u}_{k-3}^{(i+1)})^2, i > 1$ , are calculated by repeating the following procedure.

Step 1. Calculate estimates  $\hat{u}_k^{(i)}$  of  $u_k, k \in [2, N]$ , from

$$\hat{u}_k^{(i+1)} = \tilde{y}_k^{(j)} - \hat{b}_j^{(i)} \hat{u}_{k-1}^{(i)}, = 0. \quad (52)$$

Then normalise the sequence  $\hat{u}_k^{(i+1)}$  so that  $E\{\hat{u}^2\} = 1$ .

Step 2. Obtain a new estimate of  $b$  by searching for

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"No man should escape our universities without knowing how little he knows." J. Robert Oppenheimer

$$\hat{b}_j^{(i+1)} = \arg \min_{\hat{b}_j^{(i+1)} \in (-1,1)} F = \sum_{k=1}^N (\tilde{y}_k^{(j)} - \hat{b}_j^{(i+1)} \hat{u}_k^{(i+1)})^2. \quad (53)$$

The above iterations may be terminated when the difference between successive estimates is less than a prescribed tolerance.

**Lemma 8 [25]:** *The search problem in Step 2 is convex.*

**Proof:** *The claim follows from* 
$$\frac{d^2 F}{d(\hat{b}_j^{(i+1)})^2} = \sum_{k=1}^N (\hat{u}_{k-1}^{(i+1)})^2 + 5(\hat{b}_j^{(i+1)} \hat{u}_{k-2}^{(i+1)})^2 + 4(\hat{b}_j^{(i+1)} \hat{b}_j \hat{u}_{k-2}^{(i+1)})^2 + (b_j - \hat{b}_j^{(i+1)} \hat{u}_{k-3}^{(i+1)})^2.$$

A binary search method is employed within the example that is presented subsequently. Lemma 8 indicates that the searches will converge to the minimum of  $F$  within a finite number of steps. It is observed below that a reducing sequence of output estimation error variances results in a sequence of  $\hat{b}_j^2$  that is nonincreasing.

**Lemma 9:** *Suppose that there exist output estimation error sequences  $\tilde{y}_k^{(j)}$  and  $\tilde{y}_k^{(j+1)}$  such that  $E\{(\tilde{y}^{(j+1)})^2\} \leq E\{(\tilde{y}^{(j)})^2\}$ , then Procedure 1 produces  $\hat{b}_j$ ,  $\hat{b}_{j+1}$  satisfying  $\hat{b}_{j+1}^2 \leq \hat{b}_j^2$ .*

**Proof:** *It follows from (52) that Procedure 1 identify  $\hat{u}_k$  and  $\hat{b}_j$  that satisfy  $\tilde{y}_k^{(j)} = \hat{u}_k + \hat{b}_j \hat{u}_{k-1}$ , which implies*

$$\lim_{k \rightarrow \infty} E\{(\tilde{y}^{(j)})^2\} = (1 + \hat{b}_j^2) E\{\hat{u}^2\}. \quad (54)$$

*Since  $E\{\hat{u}^2\} = 1$  within Step 1 of Procedure 1, the result follows from (54) and the stated condition.*

### 5.7.7 Frequency Weighted Output Estimation

A procedure for frequency-weighted output estimation is described below.

**Procedure 2 [25]:** Assume that there exist estimates  $\hat{A}$  of  $A$  and  $\hat{\sigma}_w^2$  of  $\sigma_w^2$  which

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"All that was great in the past was ridiculed, condemned, combated, suppressed--only to emerge all the more powerfully, all the more triumphantly from the struggle." *Nikola Tesla*

are employed within the filter recursions (46) – (47) to produce unweighted output estimates  $\tilde{y}_k^{(j)}$ ,  $j = 1$ , from the measurements  $z_k$ . Frequency-weighted output estimates  $\hat{y}^{(j+1)}$  may be calculated by carrying out the following steps.

Step 1. From (53), namely, the difference between the measurement and the current output estimate, use Procedure 1 to identify  $\hat{b}_j$ .

Step 2. Realise scaled frequency-weighted output estimates (50) as

$$\hat{y}_k^{(j+1)} = (0.99(1 + \hat{b}_j))^{-1}(\hat{y}_k^{(j)} + \hat{b}_j \hat{y}_{k-1}^{(j)}) .$$

The above scaling ensures that the inverse frequency weighting transfer function's gain approaches unity at zero frequency. The 0.99 factor within the above equation sets the frequency weighting system's induced norm to less than unity which is needed in a lemma that follows.

Step 3. Optionally go to Step 2 for iterations  $j > 1$ .

The procedure may be terminated after successive estimates of  $\hat{y}_k$  have converged.

It is observed below that the above procedure leads to  $-1 < \hat{b}_j < 0$  that can result in performance benefits. Let  $||\cdot||$  and  $||\cdot||_\infty$  denote the magnitude and  $\infty$ -norm, respectively.

**Lemma 10 [25]:** An identified  $-1 < \hat{b}_j < 0$  implies (i)  $\|\tilde{y}^{(j+1)}\|_2^2 \leq \|\tilde{y}^{(j)}\|_2^2$  and (ii)  $E\{(\tilde{y}^{(j+1)})^2\} \leq E\{(\tilde{y}^{(j)})^2\}$ .

**Proof:** (i) From the definition of  $\mathcal{W}_j^{-1}$  within Step 2 of Procedure 2 and the stated condition, it follows that the  $z$ -domain frequency-weighting transfer function  $W_j(z) = 0.99(1 + \hat{b}_j)(z + \hat{b}_j)^{-1}$  is low pass and  $\|W_j(z)W_j^H(z)\|_\infty = \sup_{z \in (1, -1)} |W_j(z)W_j^H(z)| = 0.99^2 \leq 1$  for which (50) implies  $\|\tilde{y}^{(j+1)}\|_2^2 \leq \|\tilde{y}^{(j)}\|_2^2 \leq 1$ .

(ii) Since  $w$  and  $v$  are zero mean,  $\tilde{y}^{(j)}$  is zero mean,  $\|\tilde{y}^{(j)}\|_2^2 = NE\{(\tilde{y}^{(j)})^2\}$  and the claim follows from (i).

**Remark 1:** Lemmas 9 and 10 together imply that Procedure 2 produces a sequence

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“Whenever people agree with me I always feel I must be wrong.” Oscar Fingal O’Flahertie Wills Wilde

of  $\hat{b}_j^2$  that is nonincreasing. That is, iterating Step 3 of Procedure 2 results in diminishing performance benefits.

### 5.7.8 Frequency Weighting Application

It suffices to demonstrate frequency weighting with a first-order plant. In principle, a similar approach could be applied with higher order plants such as in [22], [23]. Suppose that  $A \in \mathbb{R}$ ,  $B = 1$  and  $C = 1$ . An estimate  $\hat{A}$  of  $A$  may be found by minimising the objective function  $F = \frac{1}{N-1} \sum_{k=1}^{N-1} (x_{k+1} - Ax_k)^2$ , see [23].

Setting  $\frac{\partial F}{\partial A} = 0$  yields the least-squares solution

$$\hat{A} = \left( \sum_{k=1}^{N-1} x_{k+1} x_k \right) \left( \sum_{k=1}^{N-1} (x_k)^2 \right)^{-1}. \quad (55)$$

It follows from (1) as  $k \rightarrow \infty$  that  $\sigma_x^2 = A^2 \sigma_x^2 + \sigma_w^2$ , where  $\hat{\sigma}_x^2$  is the sample variance of  $x$ . Hence an estimate of  $\hat{\sigma}_w^2$  of  $\sigma_w^2$  may be obtained from

$$\hat{\sigma}_w^2 = (1 - \hat{A}^2) \hat{\sigma}_x^2 \quad (56)$$

Since  $C = 1$ , the measurements  $z_k$  may be used in lieu of the unknown  $x_k$  within (55) and (56).

A simulation study was conducted in which measurements were generated using (1) – (3) with  $A = 0.9$ ,  $\hat{\sigma}_w^2 = 0.01$ , normally distributed realisations of  $w$  and  $v$  of length  $N = 100,000$ . Parameter estimates,  $\hat{A}$ ,  $\hat{\sigma}_w^2$  were obtained from (55) - (56) at SNRs from 0 to 5 dB. The observed root-mean-square error (RMSE) of the output estimates produced by Procedure 2 are shown in Fig. 2. It can be seen that a first-iteration frequency weighted filter (ii) improves on an unweighted filter (i) and a second iteration of frequency weighting (iii) provides a further improvement, which illustrate Lemmas 9 - 10.

It was found that repeated frequency weighting iterations provided diminishing performance improvement. For example, at 0 dB SNR,  $\hat{A} = 0.45$ ,  $\hat{\sigma}_w^2 = 0.04$  were obtained from (55) - (56), and Procedure 2 yielded  $\hat{b} = -0.06, -0.03$  and  $-0.01$  for the first, second and third iterations of frequency weighted filtering, respectively.

---

“I think anybody who doesn't think I'm smart enough to handle the job is underestimating.” *George Walker Bush*

The corresponding magnitude spectra of the inverse frequency weighting transfer functions are shown in Fig. 3. The figure shows that inverse frequency weightings are high pass and approach a short circuit as  $\hat{b}^2 \rightarrow 0$ , which is consistent with the stated interpretation of Lemmas 9 - 10 in Remark 1. Fig. 2 also demonstrates that iterated frequency-weighting provides additional performance benefits which is consistent with Lemmas 9 - 10.

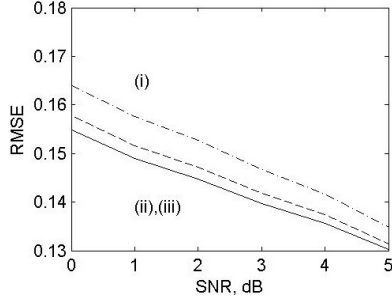


Fig. 2. RMSE for the Example: (i) unweighted filter; (ii) first iteration frequency-weighted filter; (iii) second iteration frequency weighted filter.

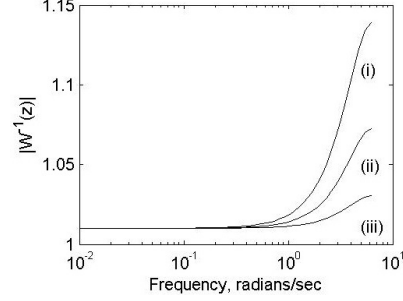


Fig. 3. Spectra of  $|W^{-1}(z)|$  for the filtering example at 0 dB SNR: (i)  $\hat{b} = -0.06$ ; (ii)  $\hat{b} = -0.03$ ; and (iii)  $\hat{b} = -0.01$ .

## 5.8 Chapter Summary

In the linear time-invariant case, it is assumed that the signals and observations can be described by  $x_{k+1} = Ax_k + Bw_k$ ,  $y_k = Cx_k$  and  $z_k = y_k + v_k$ , respectively, where the matrices  $A$ ,  $B$ ,  $C$ ,  $Q$  and  $R$  are constant. The Kalman filter for this problem is listed in Table 3. If the pair  $(A, C)$  is completely observable, the solution of the corresponding Riccati difference equation monotonically converges to the unique solution of the algebraic Riccati equation that appears in the table.

The implementation cost is lower than for time-varying problems because the gains can be calculated before running the filter. If  $|\lambda_i(A)| < 1$ ,  $i = 1$  to  $n$ , and the pair  $(A, C)$  is completely observable, then  $|\lambda_i(A - KC)| < 1$ , that is, the steady-state filter is asymptotically stable. The output estimator has the transfer function

$$H_{OE}(z) = C(I - LC)(zI - A + KC)^{-1}K + CL.$$

Since the task of solving an algebraic Riccati equation is equivalent to spectral factorisation, the transfer functions of the minimum-mean-square error and steady-state minimum-variance solutions are the same.

---

“Ideas are more powerful than guns. We would not let our enemies have guns, why should we let them have ideas.” *Joseph Stalin*

Optimal filters rely on the assumption that the model parameters and noise statistics are known, otherwise performance degradations can result. It is demonstrated above that filter estimation errors can be assumed to be generated by a first-order moving-average system. This assumed system can be identified and used to design a frequency weighting function to improve mean-square-error performance. The same remark applies to optimal smoothing.

	ASSUMPTIONS	MAIN RESULTS
Signals and system	$E\{w_k\} = E\{v_k\} = 0$ . $E\{w_k w_k^T\} = Q$ and $E\{v_k v_k^T\} = R$ are known. $A$ , $B$ and $C$ are known.	$x_{k+1} = Ax_k + Bw_k$ $y_k = Cx_k$ $z_k = y_k + v_k$
Filtered state and output estimate		$\hat{x}_{k+1/k} = (A - KC)\hat{x}_{k/k-1} + Kz_k$ $\hat{y}_{1,k/k} = (C - LC)\hat{x}_{k/k-1} + Lz_k$
Predictor gain, filter gain and algebraic Riccati equation	$Q > 0$ , $R > 0$ and $CPC^T + R_k > 0$ . The pair $(A, C)$ is observable.	$K = APC^T (CPC^T + R)^{-1}$ $L = CPC^T (CPC^T + R)^{-1}$ $P = APA^T - K(CPC^T + R)K^T + BQB^T$
Frequency weighting	Invertible frequency-weighting system $\mathcal{W} = \mathcal{W}_M \cdots \mathcal{W}_2 \mathcal{W}_1 \mathcal{W}_0$	Frequency-weighted filter $\mathcal{H} = \mathcal{W}^{-1} (I - \sigma_v^2 \{\Delta^{-H}\}_+ \Delta^{-1})$ where $\hat{y}^{(j+1)} = \mathcal{W}_j^{-1} \hat{y}^{(j)}$ , iteration $j$

Table 3. Main results for time-invariant output estimation.

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“New scientific ideas never spring from a communal body, however organized, but rather from the head of an individually inspired researcher who struggles with his problems in lonely thought and unites all his thought on one single point which is his whole world for the moment.” *Max Karl Ernst Ludwig Planck*

## 5.9 Problems

**Problem 1.** Calculate the observability matrices and comment on the observability of the following pairs.

$$(i) \ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ C = [2 \ 4]. \quad (ii) \ A = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}, \ C = [2 \ 4].$$

**Problem 2.** Generalise the proof of Lemma 1 (which addresses the unforced system  $x_{k+1} = Ax_k$  and  $y_k = Cx_k$ ) for the system  $x_{k+1} = Ax_k + Bw_k$  and  $y_k = Cx_k + Dw_k$ .

**Problem 3.** Consider the two Riccati difference equations

$$P_{t+k} = AP_{t+k-1}A^T - AP_{t+k-1}C^T(CP_{t+k-1}C^T + R)^{-1}CP_{t+k-1}A^T + BQB^T$$

$$P_{t+k+1} = AP_{t+k}A^T - AP_{t+k}C^T(CP_{t+k}C^T + R)^{-1}CP_{t+k}A^T + BQB^T.$$

Show that a Riccati difference equation for  $\bar{P}_{t+k} = P_{t+k-1} - P_{t+k}$  is given by

$$\bar{P}_{t+k+1} = \bar{A}_k \bar{P}_{t+k} \bar{A}_k^T - \bar{A}_k \bar{P}_{t+k} C^T (C \bar{P}_{t+k} C^T + \bar{R}_k)^{-1} C \bar{P}_{t+k} \bar{A}_k^T$$

where  $\bar{A}_{t+k} = A_{t+k} - A_{t+k} P_{t+k} C^T (C P_{t+k} C^T + \bar{R}_{t+k})^{-1} C$  and  $\bar{R}_{t+k} = C P_{t+k} C^T + R$ .

**Problem 4.** Suppose that measurements are generated by the single-input-single-output system  $\underline{x}_{k+1} = ax_k + w_k$ ,  $z_k = x_k + v_k$ , where  $a \in \mathbb{R}$ ,  $E\{v_k\} = 0$ ,  $E\{w_j w_k^T\} = (1-a^2)\delta_{jk}$ ,  $E\{v_j v_k^T\} = \delta_{jk}$ ,  $E\{w_j v_k^T\} = 0$ .

- Find the predicted error variance.
- Find the predictor gain.
- Verify that the one-step-ahead minimum-variance predictor is realised by

$$\hat{x}_{k+1/k} = \frac{a}{1+\sqrt{1-a^2}} \hat{x}_{k/k-1} + \frac{a\sqrt{1-a^2}}{1+\sqrt{1-a^2}} z_k.$$

- Find the filter gain.
- Write down the realisation of the minimum-variance filter.

**Problem 5.** Assuming that a system  $G$  has the realisation  $\underline{x}_{k+1} = A_k \underline{x}_k + B_k w_k$ ,  $y_k = C_k \underline{x}_k + D_k w_k$ , expand  $\Delta \Delta^H(z) = GQG(z) + R$  to obtain  $\Delta(z)$  and the optimal output estimation filter.

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“Thoughts, like fleas, jump from man to man. But they don’t bite everybody.” *Baron Stanislaw Jerzy Lec*

## 5.10 Glossary

In addition to the terms listed in Section 2.6, the notation has been used herein.

$A, B, C, D$	A linear time-invariant system is assumed to have the realisation $\underline{x}_{k+1} = Ax_k + Bw_k$ and $y_k = Cx_k + Dw_k$ in which $A, B, C, D$ are constant state space matrices of appropriate dimension.
$Q, R$	Time-invariant covariance matrices of stationary stochastic signals $w_k$ and $v_k$ , respectively.
$O$	Observability matrix.
$W$	Observability gramian.
$P$	Steady-state error covariance matrix..
$K$	Time-invariant predictor gain matrix.
$L$	Time-invariant filter gain matrix.
$\Delta(z)$	Spectral factor.
$H_{OE}(z)$	Transfer function matrix of output estimator.
$H_{IE}(z)$	Transfer function matrix of input estimator.
$\mathcal{W}$	An invertible frequency-weighting system that can be applied to frequency weight the filtered estimate.

## 5.11 References

- [1] K. Ogata, *Discrete-time Control Systems*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1987.
- [2] M. Gopal, *Modern Control Systems Theory*, New Age International Limited Publishers, New Delhi, 1993.
- [3] M. R. Opmeer and R. F. Curtain, “Linear Quadratic Gaussian Balancing for Discrete-Time Infinite-Dimensional Linear Systems”, *SIAM Journal of Control and Optimization*, vol. 43, no. 4, pp. 1196 – 1221, 2004.
- [4] S. W. Chan, G. C. Goodwin and K. S. Sin, “Convergence Properties of the Riccati Difference Equation in Optimal Filtering of Nonstabilizable Systems”, *IEEE Transactions on Automatic Control*, vol. 29, no. 2, pp. 110 – 118, Feb. 1984.
- [5] C. E. De Souza, M. R. Gevers and G. C. Goodwin, “Riccati Equations in Optimal Filtering of Nonstabilizable Systems Having Singular State Transition Matrices”, *IEEE Transactions on Automatic Control*, vol. 31, no. 9, pp. 831 – 838, Sep. 1986.
- [6] C. E. De Souza, “On Stabilizing Properties of Solutions of the Riccati Difference Equation”, *IEEE Transactions on Automatic Control*, vol. 34, no. 12, pp. 1313 – 1316, Dec. 1989.
- [7] R. R. Bitmead, M. Gevers and V. Wertz, *Adaptive Optimal Control. The thinking Man’s GPC*, Prentice Hall, New York, 1990.
- [8] R. R. Bitmead and Michel Gevers, “Riccati Difference and Differential Equations: Convergence, Monotonicity and Stability”, In S. Bittanti, A. J. Laub and J. C.

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“Nothing in life is to be feared, it is only to be understood. Now is the time to understand more, so that we may fear less.” *Marie Skłodowska Curie, winner of the 1911 Nobel Prize in Chemistry*



- Willems (Eds.), *The Riccati Equation*, Springer Verlag, 1991.
- [9] H. K. Wimmer, "Monotonicity and Maximality of Solutions of Discrete-time Algebraic Riccati Equations", *Journal of Mathematical Systems, Estimation and Control*, vol. 2, no. 2, pp. 219 – 235, 1992.
  - [10] H. K. Wimmer and M. Pavon, "A comparison theorem for matrix Riccati difference equations", *Systems and Control Letters*, vol. 19, pp. 233 – 239, 1992.
  - [11] G. A. Einicke, "Asymptotic Optimality of the Minimum Variance Fixed-Interval Smoother", *IEEE Transactions on Signal Processing*, vol. 55, no. 4, pp. 1543 – 1547, Apr. 2007.
  - [12] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1979.
  - [13] G. Freiling, G. Jank and H. Abou-Kandil, "Generalized Riccati Difference and Differential Equations", *Linear Algebra and its Applications*, vol. 241, pp. 291 – 303, 1996.
  - [14] G. Freiling and V. Ionescu, "Time-varying discrete Riccati equation: some monotonicity results", *Linear Algebra and its Applications*, vol. 286, pp. 135 – 148, 1999.
  - [15] M. J. Grimble and A. E. Sayed, "Solution of the  $H_\infty$  Optimal Linear Filtering Problem for Discrete-Time Systems", *IEEE Trans. Acoustics, Speech and Signal Proc.*, vol. 38, no. 7, pp. 1092 - 1104, Jul. 1990.
  - [16] Z. Zang, R. R. Bitmead and M. Gevers, " $H_2$  Iterative Model Refinement and Control Robustness Enhancement", *Proc. IEEE Conf. Decision and Control*, Brighton, England, pp. 279 – 284, Dec. 1991
  - [17] H. R. Pots, I. R. Petersen and A. G. Kelkar, "Robust Control of a 2D Acoustic Enclosure", *Transactions of the ASME*, vol. 125, pp. 374 – 383, 2003.
  - [18] N. Shetty and J. D. Gibson, "Perceptually Weighted Distortion Measures and the Tandem Connection of Speech Codecs", *Proc. IEEE Conf. Acoustics, Speech and Signal Processing*, Las Vegas, USA, pp. 4009 – 4012, Mar. 2008.
  - [19] C. E. de Souza, U. Shaked and M. Fu, "Robust  $H_\infty$  Filtering with Parametric Uncertainty and Deterministic Input Signal", *Proceedings 31<sup>st</sup> IEEE Conference on Decision and Control*, Tucson, Arizona, pp. 2305 – 2310, Dec. 1992.
  - [20] L. Xie, C. E. De Souza and Y. Wang, "Robust Control of Discrete Time Uncertain Dynamical Systems", *Automatica*, vol. 29, no. 4, pp. 1133 – 1137, 1993.
  - [21] G. A. Einicke, "Optimal and Robust Noncausal Filter Formulations", *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 1069 - 1077, Mar. 2006.
  - [22] G. A. Einicke, J. T. Malos, D. C. Reid and D. W. Hainsworth, "Riccati Equation and EM Algorithm Convergence for Inertial Navigation Alignment", *IEEE Transactions on Signal Processing*, vol. 57, no. 1, pp. 370 – 375, Jan. 2009.
  - [23] G. A. Einicke, G. Falco and J. T. Malos, "EM Algorithm State Matrix Estimation for Navigation", *IEEE Signal Processing Letters*, vol. 17, no. 5, pp. 437 – 440, May 2010.
  - [24] F. L. Lewis, L. Xie and D. Popa, *Optimal and Robust Estimation: With an Introduction to Stochastic Control Theory*, CRC Press Taylor & Francis Group, Boca Raton, USA, 2008.
  - [25] G. A. Einicke, "Iterative frequency-weighted filter and smoother procedures", *IEEE Signal Processing Letters*, vol. 21, iss. 12, pp. 1467 – 1470, Dec. 2014.

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"Man is but a reed, the most feeble thing in nature, but he is a thinking reed." *Blaise Pascal*

## 6. Continuous-time Smoothing

### 6.1 Introduction

The previously-described minimum-mean-square-error and minimum-variance filtering solutions operate on measurements up to the current time. If some processing delay can be tolerated then improved estimation performance can be realised through the use of smoothers. There are three state-space smoothing technique categories, namely, fixed-point, fixed-lag and fixed-interval smoothing. Fixed-point smoothing refers to estimating some linear combination of states at a previous instant in time. In the case of fixed-lag smoothing, a fixed time delay is assumed between the measurement and on-line estimation processes. Fixed-interval smoothing is for retrospective data analysis, where measurements recorded over an interval are used to obtain the improved estimates. Compared to filtering, smoothing has a higher implementation cost, as it has increased memory and calculation requirements.

A large number of smoothing solutions have been reported since Wiener's and Kalman's development of the optimal filtering results – see the early surveys [1] – [2]. The minimum-variance fixed-point and fixed-lag smoother solutions are well known. Two fixed-interval smoother solutions, namely the maximum-likelihood smoother developed by Rauch, Tung and Striebel [3], and the two-filter Fraser-Potter formula [4], have been in widespread use since the 1960s. However, the minimum-variance fixed-interval smoother is not well known. This smoother is simply a time-varying state-space generalisation of the optimal Wiener solution. It differs from the Rauch-Tung-Striebel and Fraser-Potter solutions, which may not sit well with more orthodox practitioners.

The main approaches for continuous-time fixed-point, fixed-lag and fixed-interval smoothing are canvassed here. It is assumed throughout that the underlying noise processes are zero mean and uncorrelated. Nonzero means and correlated processes can be handled using the approaches of Chapters 3 and 4. It is also assumed here that the noise statistics and state-space model parameters are known precisely. Note that techniques for estimating parameters and accommodating uncertainty are addressed subsequently.

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“Life has got a habit of not standing hitched. You got to ride it like you find it. You got to change with it. If a day goes by that don't change some of your old notions for new ones, that is just about like trying to milk a dead cow.” *Woodrow Wilson Guthrie*

Some prerequisite concepts, namely time-varying adjoint systems, backwards differential equations, Riccati equation comparison and the continuous-time maximum-likelihood method are covered in Section 6.2. Section 6.3 outlines a derivation of the fixed-point smoother by Meditch [5]. The fixed-lag smoother reported by Sage *et al* [6] and Moore [7], is the subject of Section 4. Section 5 deals with the Rauch-Tung-Striebel [3], Fraser-Potter [4] and minimum-variance fixed-interval smoother solutions [8] - [10]. As before, the approach here is to accompany the developments, where appropriate, with proofs about performance being attained. Smoothing is not a panacea for all ills. If the measurement noise is negligible then smoothing (and filtering) may be superfluous. Conversely, if measurement noise obliterates the signals then data recovery may not be possible. Therefore, estimator performance is often discussed in terms of the prevailing signal-to-noise ratio.

## 6.2 Prerequisites

### 6.2.1 Time-varying Adjoint Systems

Since fixed-interval smoothers employ backward processes, it is pertinent to introduce the adjoint of a time-varying continuous-time system. Let  $\mathcal{G}$  denote a linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (1)$$

$$y(t) = C(t)x(t) + D(t)w(t), \quad (2)$$

operating on the interval  $[0, T]$ . Let  $w$  denote the set of  $w(t)$  over all time  $t$ , that is,  $w = \{w(t), t \in [0, T]\}$ . Similarly, let  $y = \mathcal{G} w$  denote  $\{y(t), t \in [0, T]\}$ . The adjoint of  $\mathcal{G}$ , denoted by  $\mathcal{G}^H$ , is the unique linear system satisfying

$$\langle y, \mathcal{G} w \rangle = \langle \mathcal{G}^H y, w \rangle \quad (3)$$

for all  $y \in \mathbb{R}^q$  and  $w \in \mathbb{R}^p$ .

**Lemma 1:** The adjoint  $\mathcal{G}^H$  of the system  $\mathcal{G}$  described by (1) – (2), with  $x(t_0) = 0$ , having the realisation

$$\dot{\zeta}(t) = -A^T(t)\zeta(t) - C^T(t)u(t), \quad (4)$$

$$z(t) = B^T(t)\zeta(t) + D^T(t)u(t), \quad (5)$$

with  $\zeta(T) = 0$ , satisfies (3).

---

“The simple faith in progress is not a conviction belonging to strength, but one belonging to acquiescence and hence to weakness.” *Norbert Wiener*

The proof follows *mutatis mutandis* from that of Lemma 1 of Chapter 3 and is set out in [11]. The original system (1) – (2) needs to be integrated forwards in time, whereas the adjoint system (4) – (5) needs to be integrated backwards in time. Some important properties of backward systems are discussed in the next section. The simplification  $D(t) = 0$  is assumed below unless stated otherwise.

## 6.2.2 Backwards Differential Equations

The adjoint state evolution (4) is rewritten as

$$-\dot{\zeta}(t) = A^T(t)\zeta(t) + C^T(t)u(t). \quad (6)$$

The negative sign of the derivative within (6) indicates that this differential equation proceeds backwards in time. The corresponding state transition matrix is defined below.

**Lemma 2:** *The differential equation (6) has the solution*

$$\xi(t) = \Phi^H(t, t_0)\zeta(t_0) - \int_{t_0}^t \Phi^H(s, t)C^T(s)u(s)ds, \quad (7)$$

where the adjoint state transition matrix,  $\Phi^H(t, t_0)$ , satisfies

$$\dot{\Phi}^H(t, t_0) = \frac{d\Phi^H(t, t_0)}{dt} = -A^T(t)\Phi^H(t, t_0), \quad (8)$$

with boundary condition

$$\Phi^H(t, t) = I. \quad (9)$$

**Proof:** *Following the proof of Lemma 1 of Chapter 3, by differentiating (7) and substituting (4) – (5), it is easily verified that (7) is a solution of (6).*

The Lyapunov equation corresponding to (6) is described next because it is required in the development of backwards Riccati equations.

**Lemma 3:** *In respect of the backwards differential equation (6), assume that  $u(t)$  is a zero-mean white process with  $E\{u(t)u^T(\tau)\} = U(t)\delta(t - \tau)$  that is uncorrelated with  $\zeta(t_0)$ , namely,  $E\{u(t)\xi^T(t_0)\} = 0$ . Then the covariances  $P(t, \tau) =$*

*$E\{\xi(t)\xi^T(t)\}$  and  $\dot{P}(t, \tau) = \frac{dE\{\xi(t)\xi^T(t)\}}{dt}$  satisfy the Lyapunov differential equation*

---

“Progress always involves risk; you can’t steal second base and keep your foot on first base.”  
Frederick James Wilcox

$$-\dot{P}(t, \tau) = A(t)^T P(t, \tau) + P(t, \tau) A(t) - C^T(t) U(t) C(t). \quad (10)$$

**Proof:** The backwards Lyapunov differential equation (10) can be obtained by using (6) and (7) within  $\frac{dE\{\xi(t)\xi^T(t)\}}{dt} = E\{\dot{\xi}(t)\xi^T(t) + \xi(t)\dot{\xi}^T(t)\}$  (see the proof of Lemma 2 in Chapter 3).

### 6.2.3 Comparison of Riccati Equations

The following Riccati Equation comparison Theorem is required subsequently to compare the performance of filters and smoothers.

**Theorem 1 (Riccati Equation Comparison Theorem) [12], [8]:** Let  $P_1(t) \geq 0$  and  $P_2(t) \geq 0$  denote solutions of the Riccati differential equations

$$\begin{aligned} \dot{P}_1(t) = & A_1(t)P_1(t) + P_1(t)A_1^T(t) - P_1(t)S_1(t)P_1(t) \\ & + B_1(t)Q_1(t)B_1^T(t) + B(t)Q(t)B^T(t) \end{aligned} \quad (11)$$

and

$$\begin{aligned} \dot{P}_2(t) = & A_2(t)P_2(t) + P_2(t)A_2^T(t) - P_2(t)S_2(t)P_2(t) \\ & + B_2(t)Q_2(t)B_2^T(t) + B(t)Q(t)B^T(t) \end{aligned} \quad (12)$$

with  $S_1(t) = C_1^T(t)R_1^{-1}(t)C_1(t)$ ,  $S_2(t) = C_2^T(t)R_2^{-1}(t)C_2(t)$ , where  $A_1(t)$ ,  $B_1(t)$ ,  $C_1(t)$ ,  $Q_1(t) \geq 0$ ,  $R_1(t) \geq 0$ ,  $A_2(t)$ ,  $B_2(t)$ ,  $C_2(t)$ ,  $Q_2(t) \geq 0$  and  $R_2(t) \geq 0$  are of appropriate dimensions. If

- (i)  $P_1(t_0) \geq P_2(t_0)$  for a  $t_0 \geq 0$  and
- (ii)  $\begin{bmatrix} Q_1(t) & A_1(t) \\ A_1^T(t) & -S_1(t) \end{bmatrix} \geq \begin{bmatrix} Q_2(t) & A_2(t) \\ A_2^T(t) & -S_2(t) \end{bmatrix}$  for all  $t \geq t_0$ .

Then

$$P_1(t) \geq P_2(t) \quad (13)$$

for all  $t \geq t_0$ .

**Proof:** Condition (i) of the Theorem is the initial step of an induction argument. For the induction step, denote  $\dot{P}_3(t) = \dot{P}_1(t) - \dot{P}_2(t)$ ,  $P_3(t) = P_1(t) - P_2(t)$  and  $\bar{A}(t) = -A_1^T(t) + S_1(t)P_2(t) - 0.5S_1(t)P_3(t)$ . Then

---

“The price of doing the same old thing is far higher than the price of change.” William Jefferson (Bill) Clinton

$$\begin{aligned} \dot{P}_3(t) = & \bar{A}(t)P_3(t) + P_3(t)\bar{A}^T(t) \\ & + \begin{bmatrix} I & P_2(t) \end{bmatrix} \left( \begin{bmatrix} Q_1(t) & A_1(t) \\ A_1^T(t) & -S_1(t) \end{bmatrix} - \begin{bmatrix} Q_2(t) & A_2(t) \\ A_2^T(t) & -S_2(t) \end{bmatrix} \right) \begin{bmatrix} I \\ P_2(t) \end{bmatrix} \end{aligned}$$

which together with condition (ii) yields

$$\dot{P}_3(t) \geq \bar{A}(t)P_3(t) + P_3(t)\bar{A}^T(t). \quad (14)$$

Lemma 5 of Chapter 3 and (14) imply  $\dot{P}_3(t) \geq 0$  and the claim (13) follows.

## 6.2.4 The Maximum-Likelihood Method

Rauch, Tung and Streibel famously derived their fixed-interval smoother [3] using a maximum-likelihood technique which is outlined as follows. Let  $x(t) \sim \mathcal{N}(\mu, R_{xx})$  denote a continuous random variable having a Gaussian (or normal) distribution within mean  $E\{x(t)\} = \mu$  and covariance  $E\{(x(t) - \mu)(x(t) - \mu)^T\} = R_{xx}$ . The continuous-time Gaussian probability density function of  $x(t) \in \mathbb{R}^n$  is defined by

$$p(x(t)) = \frac{1}{(2\pi)^{n/2} |R_{xx}|^{1/2}} \exp \left\{ -0.5(x(t) - \mu)^T R_{xx}^{-1} (x(t) - \mu) \right\}, \quad (15)$$

in which  $|R_{xx}|$  denotes the determinant of  $R_{xx}$ . The probability that the continuous random variable  $x(t)$  with a given probability density function  $p(x(t))$  lies within an interval  $[a, b]$  is given by the likelihood function (which is also known as the cumulative distribution function)

$$P(a \leq x(t) \leq b) = \int_a^b p(x(t)) dx. \quad (16)$$

The Gaussian likelihood function for  $x(t)$  is calculated from (15) and (16) as

$$f(x(t)) = \frac{1}{(2\pi)^{n/2} |R_{xx}|^{1/2}} \int_{-\infty}^{\infty} \exp \left\{ -0.5(x(t) - \mu)^T R_{xx}^{-1} (x(t) - \mu) \right\} dx. \quad (17)$$

It is often more convenient to work with the log-probability density function

$$\log p(x(t)) = -\log (2\pi)^{n/2} |R_{xx}|^{1/2} - 0.5(x(t) - \mu)^T R_{xx}^{-1} (x(t) - \mu) \quad (18)$$

and the log-likelihood function

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“Faced with the choice between changing one’s mind and proving that there is no need to do so, almost everyone gets busy on the proof.” *John Kenneth Galbraith*

$$\log f(x(t)) = -\log (2\pi)^{n/2} |R_{xx}|^{1/2} - 0.5 \int_{-\infty}^{\infty} (x(t) - \mu)^T R_{xx}^{-1} (x - \mu) dx. \quad (19)$$

Suppose that a given record of  $x(t)$  is assumed to belong to a Gaussian distribution that is a function of an unknown quantity  $\theta$ . A statistical approach for estimating the unknown  $\theta$  is the method of maximum likelihood. This typically involves finding an estimate  $\hat{\theta}$  that either maximises the log-probability density function

$$\hat{\theta} = \arg \max_{\theta} \log p(\theta | x(t)) \quad (20)$$

or maximises the log-likelihood function

$$\hat{\theta} = \arg \max_{\theta} \log f(\theta | x(t)). \quad (21)$$

So-called maximum likelihood estimates can be found by setting either  $\frac{\partial \log p(\theta | x(t))}{\partial \theta}$  or  $\frac{\partial \log f(\theta | x(t))}{\partial \theta}$  to zero and solving for the unknown  $\theta$ .

Continuous-time maximum likelihood estimation is illustrated by the two examples that follow.

**Example 1.** Consider the first-order autoregressive system

$$\dot{x}(t) = -a_0 x(t) + w(t), \quad (22)$$

where  $\dot{x}(t) = \frac{dx(t)}{dt}$ ,  $w(t)$  is a zero-mean Gaussian process and  $a_0$  is unknown. It follows from (22) that  $\dot{x}(t) \sim \mathcal{N}(-a_0 x(t), \sigma_w^2)$ , namely,

$$f(\dot{x}(t)) = \frac{1}{(2\pi)^{n/2} \sigma_w} \int_0^T \exp \left\{ -0.5 (\dot{x}(t) + a_0 x(t))^2 \sigma_w^{-2} \right\} dt. \quad (23)$$

Taking the logarithm of both sides gives

$$\log f(\dot{x}(t)) = -\log (2\pi)^{n/2} \sigma_w - 0.5 \sigma_w^{-2} \int_0^T (\dot{x}(t) + a_0 x(t))^2 dt. \quad (24)$$

Setting  $\frac{\partial \log f(\dot{x}(t))}{\partial a_0} = 0$  results in  $\int_0^T (\dot{x}(t) + a_0 x(t)) x(t) dt = 0$  and hence

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“When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is probably wrong.” *Arthur Charles Clarke*

$$\hat{a}_0 = -\left(\int_0^T (x^2(t)dt)\right)^{-1} \int_0^T \dot{x}(t)x(t)dt. \quad (25)$$

**Example 2.** Consider the third-order autoregressive system

$$\ddot{x}(t) + a_2\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = w(t) \quad (26)$$

where  $\ddot{x}(t) = \frac{d^3x(t)}{dt^3}$  and  $\dot{x}(t) = \frac{d^2x(t)}{dt^2}$ . The above system can be written in a controllable canonical form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ 0 \\ 0 \end{bmatrix}. \quad (27)$$

Assuming  $\dot{x}_1(t) \sim \mathcal{N}(-a_2x_1(t) - a_1x_2(t) - a_0x_3(t), \sigma_w^2)$ , taking logarithms, setting to zero the partial derivatives with respect to the unknown coefficients, and rearranging yields

$$\begin{bmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = -\begin{bmatrix} \int_0^T x_3^2 dt & \int_0^T x_2x_3 dt & \int_0^T x_1x_3 dt \\ \int_0^T x_2x_3 dt & \int_0^T x_2^2 dt & \int_0^T x_2x_1 dt \\ \int_0^T x_1x_3 dt & \int_0^T x_2x_1 dt & \int_0^T x_1^2 dt \end{bmatrix}^{-1} \begin{bmatrix} \int_0^T \dot{x}_1x_3 dt \\ \int_0^T \dot{x}_1x_2 dt \\ \int_0^T \dot{x}_1x_1 dt \end{bmatrix}, \quad (28)$$

in which state time dependence is omitted for brevity.

## 6.3 Fixed-Point Smoothing

### 6.3.1 Problem Definition

In continuous-time fixed-point smoothing, it is desired to calculate state estimates at one particular time of interest,  $\tau$ ,  $0 \leq \tau \leq t$ , from measurements  $z(t)$  over the interval  $t \in [0, T]$ . For example, suppose that a continuous measurement stream of a tennis ball's trajectory is available and it is desired to determine whether it bounced within the court boundary. In this case, a fixed-point smoother could be employed to estimate the ball position at the time of the bounce from the past and future measurements.

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"Don't be afraid to take a big step if one is indicated. You can't cross a chasm in two small jumps."  
David Lloyd George



A solution for the continuous-time fixed-point smoothing problem can be developed from first principles, for example, see [5] - [6]. However, it is recognised in [13] that a simpler solution derivation follows by transforming the smoothing problem into a filtering problem that possesses an augmented state. Following the nomenclature of [14], consider an augmented state vector having two components, namely,  $x^{(a)}(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ . The first component,  $x(t) \in \mathbb{R}^n$ , is the state of the system  $\dot{x}(t) = A(t)x(t) + B(t)w(t)$  and  $y(t) = C(t)x(t)$ . The second component,  $\xi(t) \in \mathbb{R}^n$ , equals  $x(t)$  at time  $t = \tau$ , that is,  $\xi(t) = x(\tau)$ . The corresponding signal model may be written as

$$\dot{x}^{(a)}(t) = A^{(a)}(t)x^{(a)}(t) + B^{(a)}(t)w(t) \quad (29)$$

$$z(t) = C^{(a)}(t)x^{(a)}(t) + v(t), \quad (30)$$

where  $A^{(a)} = \begin{bmatrix} A(t) & 0 \\ 0 & \delta_{tr}A(t) \end{bmatrix}$ ,  $B^{(a)}(t) = \begin{bmatrix} B(t) \\ \delta_{tr}B(t) \end{bmatrix}$  and  $C^{(a)}(t) = [C(t) \ 0]$ , in which

$\delta_{tr} = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{if } t \neq \tau \end{cases}$  is the Kronecker delta function. Note that the simplifications

$A^{(a)} = \begin{bmatrix} A(t) & 0 \\ 0 & 0 \end{bmatrix}$  and  $B^{(a)}(t) = \begin{bmatrix} B(t) \\ 0 \end{bmatrix}$  arise for  $t > \tau$ . The smoothing objective is to

produce an estimate  $\hat{\xi}(t)$  of  $\xi(t)$  from the measurements  $z(t)$  over  $t \in [0, T]$ .

### 6.3.2 Solution Derivation

Employing the Kalman-Bucy filter recursions for the system (29) – (30) results in

$$\begin{aligned} \dot{\hat{x}}^{(a)}(t) &= A^{(a)}(t)\hat{x}^{(a)}(t) + K^{(a)}(t)(z(t) - C^{(a)}(t)\hat{x}(t|t)) \\ &= (A^{(a)}(t) - K^{(a)}(t)C^{(a)}(t))\hat{x}^{(a)}(t) + K^{(a)}(t)z(t), \end{aligned} \quad (31)$$

where

$$K^{(a)}(t) = P^{(a)}(t)(C^{(a)})^T(t)R^{-1}(t), \quad (32)$$

in which  $P^{(a)}(t) \in \mathbb{R}^{2n \times 2n}$  is to be found. Consider the partitioning  $K^{(a)}(t) =$

$\begin{bmatrix} K(t) \\ \underline{K}(t) \end{bmatrix}$ , then for  $t > \tau$ , (31) may be written as

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“If you don’t like change, you’re going to like irrelevance even less.” *General Eric Shinseki*

$$\begin{bmatrix} \hat{x}(t|t) \\ \hat{\xi}(t) \end{bmatrix} = \begin{bmatrix} A(t) - K(t)C(t) & 0 \\ -\underline{K}(t)C(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t|t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} K(t) \\ \underline{K}(t) \end{bmatrix} z(t). \quad (33)$$

Define the augmented error state as  $\tilde{x}^{(a)}(t) = x^{(a)}(t) - \hat{x}^{(a)}(t)$ , that is,

$$\begin{bmatrix} \tilde{x}(t|t) \\ \tilde{\xi}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} - \begin{bmatrix} \hat{x}(t|t) \\ \hat{\xi}(t) \end{bmatrix}. \quad (34)$$

Differentiating (34) and using  $z(t) = C(t)\tilde{x}(t|t) + v(t)$  gives

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}(t|t) \\ \dot{\tilde{\xi}}(t) \end{bmatrix} &= \begin{bmatrix} \dot{x}(t|t) \\ 0 \end{bmatrix} - \begin{bmatrix} \dot{\hat{x}}(t|t) \\ \dot{\hat{\xi}}(t) \end{bmatrix} \\ &= \begin{bmatrix} A(t) - K(t)C(t) & 0 \\ -\underline{K}(t)C(t) & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t|t) \\ \tilde{\xi}(t) \end{bmatrix} + \begin{bmatrix} B(t) & -K(t) \\ 0 & -\underline{K}(t) \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}. \end{aligned} \quad (35)$$

Denote  $P^{(a)}(t) = \begin{bmatrix} P(t) & \Sigma^T(t) \\ \Sigma(t) & \Omega(t) \end{bmatrix}$ , where  $P(t) = E\{[x(t) - \hat{x}(t|t)][x(t) - \hat{x}(t|t)]^T\}$ ,  $\Omega(t) = E\{[\xi(t) - \hat{\xi}(t)][\xi(t) - \hat{\xi}(t)]^T\}$  and  $\Sigma(t) = E\{[\xi(t) - \hat{\xi}(t)][x(t) - \hat{x}(t|t)]^T\}$ . Applying Lemma 2 of Chapter 3 to (35) yields the Lyapunov differential equation

$$\begin{aligned} \begin{bmatrix} \dot{P}(t) & \dot{\Sigma}^T(t) \\ \dot{\Sigma}(t) & \dot{\Omega}(t) \end{bmatrix} &= \begin{bmatrix} A(t) - K(t)C(t) & 0 \\ -\underline{K}(t)C(t) & 0 \end{bmatrix} \begin{bmatrix} P(t) & \Sigma^T(t) \\ \Sigma(t) & \Omega(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} P(t) & \Sigma^T(t) \\ \Sigma(t) & \Omega(t) \end{bmatrix} \begin{bmatrix} A^T(t) - C^T(t)K^T(t) & -C^T(t)\underline{K}^T(t) \\ 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} B(t) & -K(t) \\ 0 & -\underline{K}(t) \end{bmatrix} \begin{bmatrix} Q(t) & 0 \\ 0 & R(t) \end{bmatrix} \begin{bmatrix} B^T(t) & 0 \\ -K^T(t) & -\underline{K}^T(t) \end{bmatrix}. \end{aligned}$$

Simplifying the above differential equation yields

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t) + B(t)Q(t)B^T(t), \quad (36)$$

$$\dot{\Sigma}(t) = \Sigma(t)(A^T(t) - C^T(t)K^T(t)), \quad (37)$$

$$\dot{\Omega}(t) = -\Sigma(t)C^T(t)R^{-1}(t)C(t)\Sigma^T(t). \quad (38)$$

Equations (37) – (38) can be initialised with

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“The great virtue of my radicalism lies in the fact that I am perfectly ready, if necessary, to be radical on the conservative side” *Theodore Roosevelt*

$$\Sigma(\tau) = P(\tau) . \quad (39)$$

Thus, the fixed-point smoother estimate is given by

$$\hat{\xi}(t) = \Sigma(t)C^T(t)R^{-1}(t)(z(t) - C(t)\hat{x}(t|t)) , \quad (40)$$

which is initialised with  $\hat{\xi}(\tau) = \hat{x}(\tau)$ . Alternative derivations of (40) are presented in [5], [8], [15]. The smoother (40) and its associated error covariances (36) – (38) are also discussed in [16], [17].

### 6.3.3 Performance

It can be seen that the right-hand-side of the smoother error covariance (38) is non-positive and therefore  $\Omega(t)$  must be monotonically decreasing. That is, the smoothed estimates improve with time. However, since the right-hand-side of (36) varies inversely with  $R(t)$ , the improvement reduces with decreasing signal-to-noise ratio. It is shown below the fixed-point smoother improves on the performance of the minimum-variance filter.

**Lemma 4:** *In respect of the fixed-point smoother (40),*

$$P(t) \geq \Omega(t) . \quad (41)$$

**Proof:** *The initialisation (39) accords with condition (i) of Theorem 1. Condition (ii) of the theorem is satisfied since*

$$\begin{bmatrix} Q(t) & A(t) \\ A^T(t) & -C^T(t)R^{-1}(t)C(t) \end{bmatrix} \geq \begin{bmatrix} -C^T(t)R^{-1}(t)C(t) & 0 \\ 0 & 0 \end{bmatrix}$$

and hence the claim (41) follows.

## 6.4 Fixed-Lag Smoothing

### 6.4.1 Problem Definition

For continuous-time estimation problems, as usual, it is assumed that the observations are modelled by  $\dot{x}(t) = A(t)x(t) + B(t)w(t)$ ,  $z(t) = C(t)x(t) + v(t)$ , with  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau)$  and  $E\{v(t)v^T(\tau)\} = R(t)\delta(t-\tau)$ . In fixed-lag smoothing, it is desired to calculate state estimates at a fixed time lag behind the

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“Change will not come if we wait for some other person or some other time. We are the ones we’ve been waiting for. We are the change that we seek.” *Barack Hussein Obama II*

current measurements. That is, smoothed state estimates,  $\hat{x}(t | t + \tau)$ , are desired at time  $t$ , given data at time  $t + \tau$ , where  $\tau$  is a prescribed lag. In particular, fixed-lag smoother estimates are sought which minimise  $E\{[x(t) - \hat{x}(t | t + \tau)][x(t) - \hat{x}(t | t + \tau)]^T\}$ . It is found in [18] that the smoother yields practically all the improvement over the minimum-variance filter when the smoothing lag equals several time constants associated with the minimum-variance filter for the problem.

## 6.4.2 Solution Derivation

Previously, augmented signal models together with the application of the standard Kalman filter recursions were used to obtain the smoother results. However, as noted in [19], it is difficult to derive the optimal continuous-time fixed-lag smoother in this way because an ideal delay operator cannot easily be included within an asymptotically stable state-space system. Consequently, an alternate derivation based on that in [6] is outlined in the following. Recall that the gain of the minimum-variance filter is calculated as  $K(t) = P(t)C^T(t)R^{-1}(t)$ , where  $P(t)$  is the solution of the Riccati equation (36). Let  $\Phi(\tau, t)$  denote the transition matrix of the filter error system  $\dot{\tilde{x}}(t | t) = (A(t) - K(t)C(t))\tilde{x}(t | t) + B(t)w(t) - K(t)v(t)$ , that is,

$$\dot{\Phi}(t, s) = (A(t) - K(t)C(t))\Phi(t, s) \quad (42)$$

and  $\Phi(s, s) = I$ . It is assumed in [6], [17], [18], [20] that a smoothed estimate  $\hat{x}(t | t + \tau)$  of  $x(t)$  is obtained as

$$\hat{x}(t | t + \tau) = \hat{x}(t) + P(t)\xi(t, \tau), \quad (43)$$

where

$$\xi(t, t + \tau) = \int_t^{t+\tau} \Phi^T(\tau, t)C^T(\tau)R^{-1}(\tau)(z(\tau) - C(\tau)\hat{x}(\tau | \tau))d\tau. \quad (44)$$

The formula (43) appears in the development of fixed interval smoothers [21] - [22], in which case  $\xi(t)$  is often called an adjoint variable. From the use of Leibniz' rule, that is,

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(t, s)ds = f(t, b(t))\frac{db(t)}{dt} - f(t, a(t))\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t, s)ds,$$

---

“Change is like putting lipstick on a bulldog. The bulldog’s appearance hasn’t improved, but now it’s really angry.” *Rosbeth Moss Kanter*

it can be found that

$$\begin{aligned} \dot{\xi}(t, t+\tau) &= \Phi^T(t+\tau)C^T(t+\tau)R^{-1}(t+\tau)(z(t+\tau) - C(t+\tau)\hat{x}(t+\tau|t+\tau)) \\ &\quad - C^T(t)R^{-1}(t)(z(t) - C(t)\hat{x}(t|t)) - (A(t) - K(t)C(t))^T \xi(t, t+\tau). \end{aligned} \quad (45)$$

Differentiating (43) with respect to  $t$  gives

$$\dot{\hat{x}}(t|t+\tau) = \dot{\hat{x}}(t|t) + \dot{P}(t)\xi(t, \tau) + P(t)\dot{\xi}(t, \tau). \quad (46)$$

Substituting  $\xi(t, \tau) = P^{-1}(t)(\hat{x}(t|t+\tau) - \hat{x}(t|t))$  and expressions for  $\dot{\hat{x}}(t)$ ,  $\dot{P}(t)$ ,  $\dot{\xi}(t, t+\tau)$  into (43) yields the fixed-lag smoother differential equation

$$\begin{aligned} \dot{\hat{x}}(t|t+\tau) &= A(t)\hat{x}(t|t+\tau) + B(t)Q(t)B^T(t)P^{-1}(t)(\hat{x}(t|t+\tau) - \hat{x}(t|t)) \\ &\quad + P(t)\Phi^T(t+\tau, t)C^T(t+\tau)R^{-1}(t+\tau)(z(t+\tau) - C(t+\tau)\hat{x}(t+\tau|t+\tau)). \end{aligned} \quad (47)$$

### 6.4.3 Performance

**Lemma 5 [18]:**

$$P(t) - E\{[x(t) - \hat{x}(t|t+\tau)][x(t) - \hat{x}(t|t+\tau)]^T\} > 0. \quad (48)$$

**Proof.** It is argued from the references of [18] for the fixed-lag smoothed estimate that

$$\begin{aligned} E\{[x(t) - \hat{x}(t|t+\tau)][x(t) - \hat{x}(t|t+\tau)]^T\} \\ = P(t) - P(t) \int_t^\tau \Phi^T(s, t)C^T(s)R^{-1}(s)C(s)\Phi(s, t)ds P(t). \end{aligned} \quad (49)$$

Thus, (48) follows by inspection of (49).

That is to say, the minimum-variance filter error covariance is greater than fixed-lag smoother error covariance. It is also argued in [18] that (48) implies the error covariance decreases monotonically with the smoother lag  $\tau$ .

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“An important scientific innovation rarely makes its way by gradually winning over and converting its opponents: What does happen is that the opponents gradually die out.” *Max Karl Ernst Ludwig Planck*

## 6.5 Fixed-Interval Smoothing

### 6.5.1 Problem Definition

Many data analyses occur off-line. In medical diagnosis for example, reviews of ultra-sound or CAT scan images are delayed after the time of measurement. In principle, smoothing could be employed instead of filtering for improving the quality of an image sequence.

Fixed-lag smoothers are elegant – they can provide a small performance improvement over filters at moderate increase in implementation cost. The best performance arises when the lag is sufficiently large, at the expense of increased complexity. Thus, the designer needs to trade off performance, calculation cost and delay.

Fixed-interval smoothers are a brute-force solution for estimation problems. They provide improved performance without having to fine tune a smoothing lag, at the cost of approximately twice the filter calculation complexity. Fixed interval smoothers involve two passes. Typically, a forward process operates on the measurements. Then a backward system operates on the results of the forward process.

The plants are again assumed to have state-space realisations of the form  $\dot{x}(t) = A(t)x(t) + B(t)w(t)$  and  $y(t) = C(t)x(t) + D(t)w(t)$ . Smoothers are considered which operate on measurements  $z(t) = y(t) + v(t)$  over a fixed interval  $t \in [0, T]$ . The performance criteria depend on the quantity being estimated, viz.,

- in input estimation, the objective is to calculate a  $\hat{w}(t|T)$  that minimises  $E\{[w(t) - \hat{w}(t|T)][w(t) - \hat{w}(t|T)]^T\}$ ;
- in state estimation,  $\hat{x}(t|T)$  is calculated which achieves the minimum  $E\{[x(t) - \hat{x}(t|T)][x(t) - \hat{x}(t|T)]^T\}$ ; and
- in output estimation,  $\hat{y}(t|T)$  is produced such that  $E\{[y(t) - \hat{y}(t|T)][y(t) - \hat{y}(t|T)]^T\}$  is minimised.

This section focuses on three continuous-time fixed-interval smoother formulations; the maximum-likelihood smoother derived by Rauch, Tung and Streibel [3], the Fraser-Potter smoother [4] and a generalisation of Wiener's optimal unrealisable solution [8] – [10]. Some additional historical background to [3] – [4] is described within [1], [2], [17].

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“The soft-minded man always fears change. He feels security in the status quo, and he has an almost morbid fear of the new. For him, the greatest pain is the pain of a new idea.” *Martin Luther King Jr.*

## 6.5.2 The Maximum Likelihood Smoother

### 6.5.2.1 Solution Derivation

Rauch, Tung and Streibel [3] employed the maximum-likelihood method to develop a discrete-time smoother for state estimation and then used a limiting argument to obtain a continuous-time version. A brief outline of this derivation is set out here. Suppose that a record of filtered estimates,  $\hat{x}(\tau | \tau)$ , is available over a fixed interval  $\tau \in [0, T]$ . Let  $\hat{x}(\tau | T)$  denote smoothed state estimates at time  $0 \leq \tau \leq T$  to be evolved backwards in time from filtered states  $\hat{x}(\tau | \tau)$ . The smoother development is based on two assumptions. First, it is assumed that  $-\dot{\hat{x}}(\tau | T)$  is normally distributed with mean  $A(\tau)\hat{x}(\tau | T)$  and covariance  $B(\tau)Q(\tau)B^T(\tau)$ , that is,  $-\dot{\hat{x}}(\tau | T) \sim \mathcal{N}(A(\tau)\hat{x}(\tau | T), B(\tau)Q(\tau)B^T(\tau))$ . The probability density function of  $-\dot{\hat{x}}(\tau | T)$  is

$$p(-\dot{\hat{x}}(\tau | T) | \hat{x}(\tau | T)) = \frac{1}{(2\pi)^{n/2} |B(\tau)Q(\tau)B^T(\tau)|^{1/2}} \\ \times \exp \left\{ -0.5(-\dot{\hat{x}}(\tau | T) - A(\tau)\hat{x}(\tau | T))^T (B(\tau)Q(\tau)B^T(\tau))^{-1} (-\dot{\hat{x}}(\tau | T) - A(\tau)\hat{x}(\tau | T)) \right\}$$

Second, it is assumed that  $\hat{x}(\tau | T)$  is normally distributed with mean  $\hat{x}(\tau | \tau)$  and covariance  $P(\tau)$ , namely,  $\hat{x}(\tau | T) \sim N(\hat{x}(\tau | \tau), P(\tau))$ . The corresponding probability density function is

$$p(\hat{x}(\tau | T) | \hat{x}(\tau | \tau)) = \frac{1}{(2\pi)^{n/2} |P(\tau)|^{1/2}} \times \exp \left\{ -0.5(\hat{x}(\tau | T) - \hat{x}(\tau | \tau))^T P^{-1}(\tau) (\hat{x}(\tau | T) - \hat{x}(\tau | \tau)) \right\}.$$

From the approach of [3] and the further details in [6],

$$0 = \frac{\partial \log p(-\dot{\hat{x}}(\tau | T) | \hat{x}(\tau | T)) p(\hat{x}(\tau | T) | \hat{x}(\tau | \tau))}{\partial \hat{x}(\tau | T)} \\ = \frac{\partial \log p(-\dot{\hat{x}}(\tau | T) | \hat{x}(\tau | T))}{\partial \hat{x}(\tau | T)} + \frac{\partial \log p(\hat{x}(\tau | T) | \hat{x}(\tau | \tau))}{\partial \hat{x}(\tau | T)}$$

results in

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“If you want to truly understand something, try to change it.” *Kurt Lewin*

$$0 = \frac{\partial(-\dot{\hat{x}}(\tau | T) - A(\tau)\hat{x}(\tau | T))^T}{\partial\hat{x}(\tau | T)} \\ \times (B(\tau)Q(\tau)B^T(\tau))^{-1}(-\dot{\hat{x}}(\tau | T) - A(\tau)\hat{x}(\tau | T)) + P^{-1}(\tau)(\hat{x}(t | T) - \hat{x}(\tau | \tau)).$$

Hence, the solution is given by

$$-\dot{\hat{x}}(\tau | T) = A(\tau)\hat{x}(\tau | T) + G(\tau)(\hat{x}(\tau | T) - \hat{x}(\tau | \tau)), \quad (50)$$

where

$$G(\tau) = -B(\tau)Q(\tau)B^T(\tau) \frac{\partial(-\dot{\hat{x}}(\tau | T) - A(\tau)\hat{x}(\tau | T))^T}{\partial\hat{x}(\tau | T)} P^{-1}(\tau) \quad (51)$$

is the smoother gain. Suppose that  $\hat{x}(\tau | T)$ ,  $A(\tau)$ ,  $B(\tau)$ ,  $Q(\tau)$ ,  $P^{-1}(\tau)$  are sampled at integer  $k$  multiples of  $T_s$  and are constant during the sampling interval. Using the Euler approximation  $-\dot{\hat{x}}(kT_s | T) = \frac{\hat{x}((k-1)T_s | T) - \hat{x}(kT_s | T)}{T_s}$ , the sampled gain may be written as

$$G(kT_s) = B(kT_s)T_s^{-1}Q(kT_s)B^T(kT_s)(I + AT_s)P^{-1}(kT_s). \quad (52)$$

Recognising that  $T_s^{-1}Q(kT_s) = Q(\tau)$ , see [23], and taking the limit as  $T_s \rightarrow 0$  and yields

$$G(\tau) = B(\tau)Q(\tau)B^T(\tau)P^{-1}(\tau). \quad (53)$$

To summarise, the above fixed-interval smoother is realised by the following two-pass procedure.

- (i) In the first pass, the (forward) Kalman-Bucy filter operates on measurements  $z(\tau)$  to obtain state estimates  $\hat{x}(\tau | \tau)$ .
- (ii) In the second pass, the differential equation (50) operates on the filtered state estimates  $\hat{x}(\tau | \tau)$  to obtain smoothed state estimates  $\hat{x}(\tau | T)$ . Equation (50) is integrated backwards in time from the initial condition  $\hat{x}(\tau | T) = \hat{x}(\tau | \tau)$  at  $\tau = T$ .

Alternative derivations of this smoother appear in [6], [20], [23], [24].

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“There is a certain relief in change, even though it be from bad to worse. As I have often found in travelling in a stagecoach that it is often a comfort to shift one’s position, and be bruised in a new place.” *Washington Irving*



### 6.5.2.2 Alternative Form

For the purpose of developing an alternate form of the above smoother found in the literature, consider a fictitious forward version of (50), namely,

$$\begin{aligned}\dot{\hat{x}}(t|T) &= A(t)\hat{x}(t|T) + B(t)Q(t)B^T(t)P^{-1}(t)(\hat{x}(t|T) - \hat{x}(t|t)) \\ &= A(t)\hat{x}(t|T) + B(t)Q(t)B^T(t)\xi(t|T),\end{aligned}\quad (54)$$

where

$$\xi(t|T) = P^{-1}(t)(\hat{x}(t|T) - \hat{x}(t|t)) \quad (55)$$

is an auxiliary variable. An expression for the evolution of  $\xi(t|T)$  is now developed. Writing (55) as

$$\hat{x}(t|T) = \hat{x}(t|t) + P(t)\xi(t|T) \quad (56)$$

and taking the time differential results in

$$\dot{\hat{x}}(t|T) = \dot{\hat{x}}(t|t) + \dot{P}(t)\xi(t|T) + P(t)\dot{\xi}(t|T). \quad (57)$$

Substituting  $\dot{\hat{x}}(t|t) = A(t)\hat{x}(t|t) + P(t)C^T(t)R^{-1}(z(t) - C(t)\hat{x}(t|t))$  into (57) yields

$$\begin{aligned}P(t)\dot{\xi}(t|T) &= P(t)C^T(t)R^{-1}C(t) - P(t)C^T(t)R^{-1}z(t) \\ &\quad + A(t)P(t)\xi(t) + B(t)Q(t)B^T(t) - \dot{P}(t)\xi(t).\end{aligned}\quad (58)$$

Using  $\hat{x}(t|t) = \hat{x}(t|T) - P(t)\xi(t|T)$ ,  $-P(t)A^T(t) = A(t)P(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t)\xi(t|T) + B(t)Q(t)B^T(t) - \dot{P}(t)$  within (58) and rearranging gives

$$-\dot{\xi}(t|T) = -C^T(t)R^{-1}(t)C(t)\hat{x}(t|T) + A^T(t)\xi(t) - C^T(t)R^{-1}(t)z(t). \quad (59)$$

The filter (54) and smoother (57) may be collected together as

$$\begin{bmatrix} \dot{\hat{x}}(t|T) \\ -\dot{\xi}(t|T) \end{bmatrix} = \begin{bmatrix} A(t) & B(t)Q(t)B^T(t) \\ -C^T(t)R^{-1}(t)C(t) & A^T(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t|T) \\ \xi(t|T) \end{bmatrix} + \begin{bmatrix} 0 \\ C^T(t)R^{-1}(t)z(t) \end{bmatrix} \quad (60)$$

---

“Discontent is the first step in the progress of a man or a nation.” *Oscar Fingal O’Flahertie Wills Wilde*

Equation (60) is known as the Hamiltonian form of the Rauch-Tung-Striebel smoother [17].

### 6.5.2.3 Performance

In order to develop an expression for the smoothed error state, consider the backwards signal model

$$-\dot{x}(\tau) = A(\tau)x(\tau) + B(\tau)w(\tau). \quad (61)$$

Subtracting (50) from (61) results in

$$\begin{aligned} -\dot{x}(\tau) + \dot{\hat{x}}(\tau | T) &= (A(\tau) + G(\tau))(x(\tau) - \hat{x}(\tau | T)) \\ &\quad - G(\tau)(x(\tau) - \hat{x}(\tau | \tau)) + B(\tau)w(\tau). \end{aligned} \quad (62)$$

Let  $\tilde{x}(\tau | T) = x(\tau) - \hat{x}(\tau | T)$  denote the smoothed error state and  $\tilde{x}(\tau | \tau) = x(\tau) - \hat{x}(\tau | \tau)$  denote the filtered error state. Then the differential equation (62) can simply be written as

$$-\dot{\tilde{x}}(\tau | T) = (A(\tau) + G(\tau))(\tilde{x}(\tau | T) - G(\tau)\tilde{x}(\tau | \tau) + B(\tau)w(\tau)), \quad (63)$$

where  $-\dot{\tilde{x}}(\tau | T) = -(\dot{\hat{x}}(\tau | T) - \dot{\hat{x}}(\tau))$ . Applying Lemma 3 to (63) and using  $E\{\hat{x}(\tau | \tau), w^T(\tau)\} = 0$  gives

$$-\dot{\Sigma}(\tau | T) = (A(\tau) + G(\tau))\Sigma(\tau | T) + \Sigma(\tau | T)(A(\tau) + G(\tau))^T - B(\tau)Q(\tau)B^T(\tau), \quad (64)$$

where  $\Sigma(\tau | T) = E\{\hat{x}(\tau | T), \hat{x}^T(\tau | T)\}$  is the smoother error covariance and  $\dot{\Sigma}(\tau | T) = \frac{d\Sigma(\tau | T)}{d\tau}$ . The smoother error covariance differential equation (64) is solved backwards in time from the initial condition

$$\Sigma(\tau | T) = P(t | t) \quad (65)$$

at  $t = T$ , where  $P(t | t)$  is the solution of the Riccati differential equation

$$\begin{aligned} \dot{P}(t) &= (A(t) - K(t)C(t))P(t) + P(t)(A^T(t) - C^T(t)K^T(t)) \\ &\quad + K(t)R(t)K^T(t) + B(t)Q(t)B^T(t) \\ &= A(t)P(t) + P(t)A^T(t) - K(t)R(t)K^T(t) + B(t)Q(t)B^T(t). \end{aligned} \quad (66)$$

---

“That which comes into the world to disturb nothing deserves neither respect nor patience.” *Rene Char*

It is shown below that this smoother outperforms the minimum-variance filter. For the purpose of comparing the solutions of forward Riccati equations, consider a fictitious forward version of (64), namely,

$$\dot{\Sigma}(t | T) = (A(t) + G(t))\Sigma(t | T) + \Sigma(t | T)(A(t) + G(t))^T - B(t)Q(t)B^T(t) \quad (67)$$

initialised with

$$\Sigma(t_0 | T) = P(t_0 | t_0) > 0. \quad (68)$$

**Lemma 6:** *In respect of the fixed-interval smoother (50),*

$$P(t | t) \geq \Sigma(t | T). \quad (69)$$

**Proof:** *The initialisation (68) satisfies condition (i) of Theorem 1. Condition (ii) of the theorem is met since*

$$\begin{bmatrix} B(t)Q(t)B^T(t) + K(t)R(t)K^T(t) & A(t) - K(t)C(t) \\ A^T(t) - C^T(t)K^T(t) & 0 \end{bmatrix} \geq \begin{bmatrix} -B(t)Q(t)B^T(t) & A(t) + G(t) \\ A^T(t) + G^T(t) & 0 \end{bmatrix}$$

for all  $t \geq t_0$  and hence the claim (69) follows.

### 6.5.3 The Fraser-Potter Smoother

The Central Limit Theorem states that the mean of a sufficiently large sample of independent identically distributed random variables will be approximately normally distributed [25]. The same is true of partial sums of random variables. The Central Limit Theorem is illustrated by the first part of the following lemma. A useful generalisation appears in the second part of the lemma.

**Lemma 7:** *Suppose that  $y_1, y_2, \dots, y_n$  are independent random variables and  $W_1, W_2, \dots, W_n$  are independent positive definite weighting matrices. Let  $\mu = E\{y\}$ ,  $u = y_1 + y_2 + \dots + y_n$  and*

$$v = (W_1 y_1 + W_2 y_2 + \dots + W_n y_n) (W_1 + W_2 + \dots + W_n)^{-1}. \quad (70)$$

- (i) *If  $y_i \sim \mathcal{N}(\mu, R)$ ,  $i = 1$  to  $n$ , then  $u \sim \mathcal{N}(n\mu, nR)$ ;*
- (ii) *If  $y_i \sim \mathcal{N}(0, I)$ ,  $i = 1$  to  $n$ , then  $v \sim \mathcal{N}(0, I)$ .*

**Proof:**

- (i)  $E\{u\} = E\{y_1\} + E\{y_2\} + \dots + E\{y_n\} = n\mu$ .  $E\{(u - \mu)(u - \mu)^T\} = E\{(y_1 - \mu)(y_1 - \mu)^T\} + E\{(y_2 - \mu)(y_2 - \mu)^T\} + \dots + E\{(y_n - \mu)(y_n - \mu)^T\} = nR$ .

---

“Today every invention is received with a cry of triumph which soon turns into a cry of fear.” *Bertolt Brecht*

$$\begin{aligned}
 (ii) \quad E\{v\} &= W_1(W_1 + W_2 + \dots + W_n)^{-1}E\{y_1\} + W_2(W_1 + W_2 + \dots + W_n)^{-1}E\{y_2\} \\
 &+ \dots + W_n(W_1 + W_2 + \dots + W_n)^{-1}E\{y_n\} = 0. \quad E\{vv^T\} = E\{(W_1^T + W_2^T + \dots \\
 &+ W_n^T)^{-1}W_1^T y_1^T y_1 W_1(W_1 + W_2 + \dots + W_n)^{-1}\} + E\{(W_1^T + W_2^T + \dots + \\
 &W_n^T)^{-1}W_2^T y_2^T y_2 W_2(W_1 + W_2 + \dots + W_n)^{-1}\} + \dots + E\{(W_1^T + W_2^T + \dots + \\
 &W_n^T)^{-1}W_n^T y_n^T y_n W_n(W_1 + W_2 + \dots + W_n)^{-1}\} = (W_1^T + W_2^T + \dots + \\
 &W_n^T)^{-1}(W_1^T W_1 + W_2^T W_2 + \dots + W_n^T W_n)(W_1 + W_2 + \dots + W_n)^{-1} = I.
 \end{aligned}$$

Fraser and Potter reported a smoother in 1969 [4] that combined state estimates from forward and backward filters using a formula similar to (70) truncated at  $n = 2$ . The inverses of the forward and backward error covariances, which are indicative of the quality of the respective estimates, were used as weighting matrices. The combined filter and Fraser-Potter smoother equations are

$$\dot{\hat{x}}(t|t) = A(t)\hat{x}(t|t) + P(t|t)C^T(t)R^{-1}(t)(z(t) - C(t)\hat{x}(t|t)), \quad (71)$$

$$-\dot{\hat{\xi}}(t|t) = A(t)\hat{\xi}(t|t) + \Sigma(t|t)C^T(t)R^{-1}(t)(z(t) - C(t)\hat{\xi}(t|t)), \quad (72)$$

$$\hat{x}(t|T) = (P^{-1}(t|t) + \Sigma^{-1}(t|t))^{-1}(P^{-1}(t|t)\hat{x}(t|t) + \Sigma^{-1}(t|t)\hat{\xi}(t|t)), \quad (73)$$

where  $P(t|t)$  is the solution of the forward Riccati equation  $\dot{P}(t|t) = A(t)P(t|t) + P(t|t)A^T(t) - P(t|t)C^T(t)R^{-1}(t)C(t)P(t|t) + B(t)Q(t)B^T(t)$  and  $\Sigma(t|t)$  is the solution of the backward Riccati equation  $-\dot{\Sigma}(t|t) = A(t)\Sigma(t|t) + \Sigma(t|t)A^T(t) - \Sigma(t|t)C^T(t)R^{-1}(t)C(t)\Sigma(t|t) + B(t)Q(t)B^T(t)$ .

It can be seen from (72) that the backward state estimates,  $\hat{\xi}(t)$ , are obtained by simply running a Kalman filter over the time-reversed measurements. Fraser and Potter's approach is pragmatic: when the data is noisy, a linear combination of two filtered estimates is likely to be better than one filter alone. However, this two-filter approach to smoothing is *ad hoc* and is not a minimum-mean-square-error design.

## 6.5.4 The Minimum-Variance Smoother

### 6.5.4.1 Problem Definition

The previously described smoothers are focussed on state estimation. A different signal estimation problem shown in Fig. 1 is considered here. Suppose that observations  $z = y_2 + v$  are available, where  $y_2 = \mathcal{G}_2 w$  is the output of a linear time-varying system and  $v$  is measurement noise. A solution  $\mathcal{H}$  is desired which

---

"If there is dissatisfaction with the status quo, good. If there is ferment, so much the better. If there is restlessness, I am pleased. Then let there be ideas, and hard thought, and hard work." *Hubert Horatio Humphrey*.

produces estimates  $\hat{y}_1$  of a second reference system  $y_1 = \mathcal{G}_1 w$  in such a way to meet a performance objective. Let  $\tilde{y} = y_1 - \hat{y}_1$  denote the output estimation error. The optimum minimum-variance filter can be obtained by finding the solution that minimises  $\|\tilde{y}\tilde{y}^T\|_2$ . Here, in the case of smoothing, the performance objective is to minimise  $\|\tilde{y}\tilde{y}^H\|_2$ .

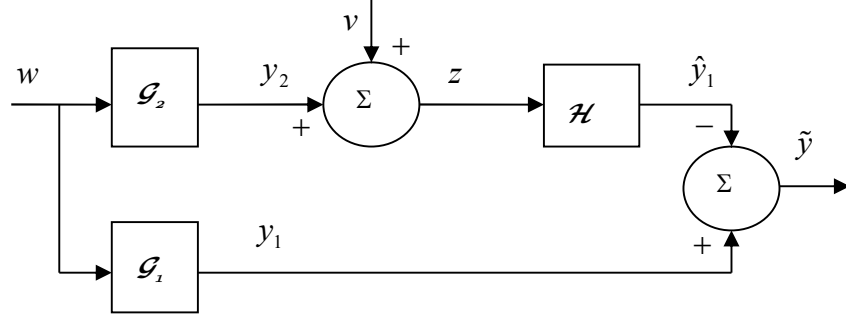


Fig. 1. The general estimation problem. The objective is to produce estimates  $\hat{y}_1$  of  $y_1$  from measurements  $z$ .

#### 6.5.4.2 Optimal Unrealisable Solutions

The minimum-variance smoother is a more recent innovation [8] - [10] and arises by generalising Wiener's optimal noncausal solution for the above time-varying problem. The solution is obtained using the same completing-the-squares technique that was previously employed in the frequency domain (see Chapters 1 and 2). It can be seen from Fig. 1 that the output estimation error is generated by  $\tilde{y} = \mathcal{R}_{yi} i$ , where

$$\mathcal{R}_{yi} = -[\mathcal{H} \quad \mathcal{H}\mathcal{G}_2 - \mathcal{G}_1] \quad (74)$$

is a linear system that operates on the inputs  $i = \begin{bmatrix} v \\ w \end{bmatrix}$ .

Consider the factorisation

$$\Delta\Delta^H = \mathcal{G}_2 Q \mathcal{G}_2^H + R, \quad (75)$$

---

"Restlessness and discontent are the first necessities of progress. Show me a thoroughly satisfied man and I will show you a failure." *Thomas Alva Edison*

in which the time-dependence of  $Q(t)$  and  $R(t)$  is omitted for notational brevity. Suppose that  $\Delta$  is causal, namely  $\Delta$  and its inverse,  $\Delta^{-1}$ , are bounded systems that proceed forward in time. The system  $\Delta$  is known as a Wiener-Hopf factor.

**Lemma 8:** Assume that the Wiener-Hopf factor inverse,  $\Delta^{-1}$ , exists over  $t \in [0, T]$ . Then the smoother solution

$$\begin{aligned}\mathcal{H} &= \mathcal{G}_1 Q \mathcal{G}_2^H \Delta^{-H} \Delta^{-1} \\ &= \mathcal{G}_1 Q \mathcal{G}_2^H (\Delta \Delta^H)^{-1} \\ &= \mathcal{G}_1 Q \mathcal{G}_2^H (\mathcal{G}_2 Q \mathcal{G}_2^H + R)^{-1}.\end{aligned}\tag{76}$$

minimises  $\|\tilde{y} \tilde{y}^H\|_2 = \|\mathcal{R}_{\tilde{y}} \mathcal{R}_{\tilde{y}}^H\|_2$ .

**Proof:** It follows from (74) that  $\mathcal{R}_{\tilde{y}} \mathcal{R}_{\tilde{y}}^H = \mathcal{G}_1 Q \mathcal{G}_2^H - \mathcal{G}_1 Q \mathcal{G}_2^H \mathcal{H}^H - \mathcal{H} \mathcal{G}_2 Q \mathcal{G}_1^H + \mathcal{H} \Delta \Delta^H \mathcal{H}^H$ . Completing the square leads to  $\mathcal{R}_{\tilde{y}} \mathcal{R}_{\tilde{y}}^H = \mathcal{R}_{\tilde{y}1} \mathcal{R}_{\tilde{y}1}^H + \mathcal{R}_{\tilde{y}2} \mathcal{R}_{\tilde{y}2}^H$ , where

$$\mathcal{R}_{\tilde{y}2} \mathcal{R}_{\tilde{y}2}^H = (\mathcal{H} \Delta - \mathcal{G}_1 Q \mathcal{G}_2^H \Delta^{-H}) (\mathcal{H} \Delta - \mathcal{G}_1 Q \mathcal{G}_2^H \Delta^{-H})^H \tag{77}$$

and

$$\mathcal{R}_{\tilde{y}1} \mathcal{R}_{\tilde{y}1}^H = \mathcal{G}_1 Q \mathcal{G}_1^H - \mathcal{G}_1 Q \mathcal{G}_2^H (\Delta \Delta^H)^{-1} \mathcal{G}_2 Q \mathcal{G}_1^H. \tag{78}$$

By inspection of (77), the solution (76) achieves

$$\|\mathcal{R}_{\tilde{y}2} \mathcal{R}_{\tilde{y}2}^H\|_2 = 0. \tag{79}$$

Since  $\|\mathcal{R}_{\tilde{y}1} \mathcal{R}_{\tilde{y}1}^H\|_2$  excludes the estimator solution  $\mathcal{H}$ , this quantity defines the lower bound for  $\|\mathcal{R}_{\tilde{y}} \mathcal{R}_{\tilde{y}}^H\|_2$ .

**Example 3.** Consider the output estimation case where  $\mathcal{G}_1 = \mathcal{G}_2$  and

$$\mathcal{H}_{OE} = \mathcal{G}_2 Q \mathcal{G}_2^H (\mathcal{G}_2 Q \mathcal{G}_2^H + R)^{-1}, \tag{80}$$

which is of order  $n^4$  complexity. Using  $\mathcal{G}_2 Q \mathcal{G}_2^H = \Delta \Delta^H - R$  leads to the  $n^2$ -order solution

$$\mathcal{H}_{OE} = I - R(\Delta \Delta^H)^{-1}. \tag{81}$$

---

“Whatever has been done before will be done again. There is nothing new under the sun.” *Ecclesiastes 1:9*

It is interesting to note from (81) and

$$\mathcal{R}_{yi1} \mathcal{R}_{yi1}^H = \mathcal{G}_2 \mathcal{Q} \mathcal{G}_2^H - \mathcal{G}_2 \mathcal{Q} \mathcal{G}_2^H (\mathcal{G}_2 \mathcal{Q} \mathcal{G}_2^H - R)^{-1} \mathcal{G}_2 \mathcal{Q} \mathcal{G}_2^H \quad (82)$$

that  $\lim_{R \rightarrow 0} \mathcal{H} = I$  and  $\lim_{R \rightarrow 0} \mathcal{R}_{yi1} \mathcal{R}_{yi1}^H = 0$ . That is, output estimation is superfluous when measurement noise is absent. Let  $\{\mathcal{R}_{yi} \mathcal{R}_{yi}^H\}_+ = \{\mathcal{R}_{yi1} \mathcal{R}_{yi1}^H\}_+ + \{\mathcal{R}_{yi2} \mathcal{R}_{yi2}^H\}_+$  denote the causal part of  $\mathcal{R}_{yi} \mathcal{R}_{yi}^H$ . It is shown below that minimum-variance filter solution can be found using the above completing-the squares technique and taking causal parts.

**Lemma 9:** *The filter solution*

$$\begin{aligned} \{\mathcal{H}\}_+ &= \{\mathcal{G}_1 \mathcal{Q} \mathcal{G}_2^H \Delta^{-H} \Delta^{-1}\}_+ \\ &= \{\mathcal{G}_1 \mathcal{Q} \mathcal{G}_2^H \Delta^{-H}\}_+ \Delta^{-1} \end{aligned} \quad (83)$$

minimises  $\|\{\tilde{\mathcal{Y}}^H\}_+\|_2 = \|\{\mathcal{R}_{yi} \mathcal{R}_{yi}^H\}_+\|_2$ , provided that the inverses exist.

**Proof:** *It follows from (77) that*

$$\{\mathcal{R}_{yi2} \mathcal{R}_{yi2}^H\}_+ = \{(\mathcal{H} \Delta - \mathcal{G}_1 \mathcal{Q} \mathcal{G}_2^H \Delta^{-H})(\mathcal{H} \Delta - \mathcal{G}_1 \mathcal{Q} \mathcal{G}_2^H \Delta^{-H})^H\}_+. \quad (84)$$

By inspection of (84), the solution (83) achieves

$$\|\{\mathcal{R}_{yi2} \mathcal{R}_{yi2}^H\}_+\|_2 = 0. \quad (85)$$

It is worth pausing at this juncture to comment on the significance of the above results.

- The formulation (76) is an optimal solution for the time-varying smoother problem since it can be seen from (79) that it achieves the best-possible (minimum-error-variance) performance.
- Similarly, (83) is termed an optimal solution because it achieves the best-possible filter performance (85).
- By inspection of (79) and (85) it follows that the minimum-variance smoother outperforms the minimum-variance filter.
- In general, these optimal solutions are not very practical because of the difficulty in realising an exact Wiener-Hopf factor.

Practical smoother (and filter) solutions that make use of an approximate Wiener-Hopf factor are described below.

---

“There is nothing more difficult to take in hand, more perilous to conduct, or more uncertain in its success, than to take the lead in the introduction of a new order of things.” *Niccolo Di Bernado dei Machiavelli*

### 6.5.4.3 Optimal Realisable Solutions

#### Output Estimation

The Wiener-Hopf factor is modelled on the structure of the spectral factor which is described Section 3.4.4. Suppose that  $R(t) > 0$  for all  $t \in [0, T]$  and there exist  $R^{1/2}(t) > 0$  such that  $R(t) = R^{1/2}(t) R^{1/2}(t)$ . An approximate Wiener-Hopf factor,  $\hat{\Delta}$ , is defined by the system

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} A(t) & K(t)R^{1/2}(t) \\ C(t) & R^{1/2}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad (86)$$

where  $K(t) = P(t)C^T(t)R^{-1}(t)$  is the Kalman gain in which  $P(t)$  is the solution of the Riccati differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t) + B(t)Q(t)B^T(t). \quad (87)$$

The output estimation smoother (81) can be approximated as

$$\begin{aligned} \mathcal{H}_{OE} &= I - R(\hat{\Delta}\hat{\Delta}^H)^{-1} \\ &= I - R\hat{\Delta}^{-H}\hat{\Delta}^{-1}. \end{aligned} \quad (88)$$

An approximate Wiener-Hopf factor inverse,  $\hat{\Delta}^{-1}$ , within (88) is obtained from (86) and the Matrix Inversion Lemma, namely,

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \alpha(t) \end{bmatrix} = \begin{bmatrix} A(t) - K(t)C(t) & K(t) \\ -R^{-1/2}(t)C(t) & R^{-1/2}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad (89)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is an estimate of the state within  $\hat{\Delta}^{-1}$ . From Lemma 1, the adjoint of  $\hat{\Delta}^{-1}$ , which is denoted by  $\hat{\Delta}^{-H}$ , has the realisation

$$\begin{bmatrix} -\dot{\xi}(t) \\ \beta(t) \end{bmatrix} = \begin{bmatrix} A^T(t) - C^T(t)K^T(t) & -C^T(t)R^{-1/2}(t) \\ K^T(t) & R^{-1/2}(t) \end{bmatrix} \begin{bmatrix} \xi(t) \\ \alpha(t) \end{bmatrix}. \quad (90)$$

where  $\xi(t) \in \mathbb{R}^p$  is an estimate of the state within  $\hat{\Delta}^{-H}$ . Thus, the smoother (88) is realised by (89), (90) and

---

“If I have a thousand ideas and only one turns out to be good, I am satisfied.” *Alfred Bernhard Nobel*



$$\hat{y}(t | T) = z(t) - R(t)\beta(t). \quad (91)$$

**Procedure 1.** The above output estimator can be implemented via the following three steps.

- Step 1. Operate  $\hat{\Delta}^{-1}$  on the measurements  $z(t)$  using (89) to obtain  $\alpha(t)$ .
- Step 2. In lieu of the adjoint system (90), operate (89) on the time-reversed transpose of  $\alpha(t)$ . Then take the time-reversed transpose of the result to obtain  $\beta(t)$ .
- Step 3. Calculate the smoothed output estimate from (91).

**Example 4.** Consider an estimation problem parameterised by  $a = -1$ ,  $b = \sqrt{2}$ ,  $c = 1$ ,  $d = 0$ ,  $\sigma_w^2 = \sigma_v^2 = 1$ , which leads to  $p = k = \sqrt{3} - 1$  [26]. Smoothed output estimates may be obtained by evolving

$$\dot{\hat{x}}(t) = \sqrt{3}\hat{x}(t) + \sqrt{3}z(t), \quad \alpha(t) = -\hat{x}(t) + z(t),$$

time-reversing the  $\alpha(t)$  and evolving

$$\dot{\xi}(t) = \sqrt{3}\xi(t) + \sqrt{3}\alpha(t), \quad \beta(t) = -\xi(t) + \alpha(t),$$

then time-reversing  $\beta(t)$  and calculating

$$\hat{y}(t | T) = z(t) - \beta(t).$$

## Filtering

The causal part  $\{\mathcal{H}_{OE}\}_+$  of the minimum-variance smoother (88) is given by

$$\begin{aligned} \{\mathcal{H}_{OE}\}_+ &= I - R\{\hat{\Delta}^{-H}\}_+ \hat{\Delta}^{-1} \\ &= I - RR^{-1/2} \hat{\Delta}^{-1} \\ &= I - R^{1/2} \hat{\Delta}^{-1}. \end{aligned} \quad (92)$$

Employing (89) within (92) leads to the standard minimum-variance filter, namely,

$$\dot{\hat{x}}(t | t) = (A(t) - K(t)C(t))\hat{x}(t | t) + K(t)z(t) \quad (93)$$

$$\hat{y}(t | t) = C(t)\hat{x}(t | t). \quad (94)$$

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“Ten geographers who think the world is flat will tend to reinforce each other’s errors....Only a sailor can set them straight.” *John Ralston Saul*

## Input Estimation

As discussed in Chapters 1 and 2, input estimates can be found using  $\mathcal{G}_1 = I$ , and substituting  $\hat{\Delta}$  for  $\Delta$  within (76) yields the solution

$$\mathcal{H}_{IE} = Q\mathcal{G}_2^{-1}(\hat{\Delta}\hat{\Delta}^H)^{-1} = Q\mathcal{G}_2^{-1}\hat{\Delta}^{-H}\hat{\Delta}^{-1}. \quad (95)$$

As expected, the low-measurement-noise-asymptote of this equaliser is given by  $\lim_{R \rightarrow 0} \mathcal{H}_{IE} = \mathcal{G}_2^{-1}$ . That is, at high signal-to-noise-ratios the equaliser approaches  $\mathcal{G}_2^{-1}$ , provided the inverse exists.

The development of a differential equation for the smoothed input estimate,  $\hat{w}(t|T)$ , makes use of the following formula [27] for the cascade of two systems.

Suppose that two linear systems  $\mathcal{G}_1$  and  $\mathcal{G}_2$  have state-space parameters

$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$  and  $\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$ , respectively. Then  $\mathcal{G}_2\mathcal{G}_1$  is parameterised by

$\begin{bmatrix} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ D_2C_1 & C_2 & D_2D_1 \end{bmatrix}$ . It follows that  $\hat{w}(t|T) = Q\mathcal{G}^H\hat{\Delta}^{-H}\alpha(t)$  is realised by

$$\begin{bmatrix} -\dot{\xi}(t) \\ -\dot{\gamma}(t) \\ -\dot{\hat{w}}(t|T) \end{bmatrix} = \begin{bmatrix} A^T(t) - C^T(t)K^T(t) & 0 & -C^T(t)R^{-1/2}(t) \\ -C^T(t)K^T(t) & A^T(t) & C^T(t)R^{-1/2}(t) \\ Q(t)D^T(t)K^T(t) & Q(t)B^T(t) & Q(t)D^T(t)R^{-1/2}(t) \end{bmatrix} \begin{bmatrix} \xi(t) \\ \gamma(t) \\ \alpha(t) \end{bmatrix}. \quad (96)$$

in which  $\gamma(t) \in \mathbb{R}^n$  is an auxiliary state.

**Procedure 2.** Input estimates can be calculated via the following two steps.

- Step 1. Operate  $\hat{\Delta}^{-1}$  on the measurements  $z(t)$  using (89) to obtain  $\alpha(t)$ .
- Step 2. In lieu of (96), operate the adjoint of (96) on the time-reversed transpose of  $\alpha(t)$ . Then take the time-reversed transpose of the result.

## State Estimation

Smoothed state estimates can be obtained by defining the reference system  $\mathcal{G}_1$  within (76) as

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“In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual.” *Galileo Galilei*

$$\dot{\hat{x}}(t | T) = A(t)\hat{x}(t | T) + B(t)\hat{w}(t | T). \quad (97)$$

That is, a smoother for state estimation is given by (89), (96) and (97). In frequency-domain estimation problems, minimum-order solutions are found by exploiting pole-zero cancellations, see Example 13 of Chapter 1. Here in the time-domain, (89), (96), (97) is not a minimum-order solution and some numerical model order reduction may be required.

Suppose that  $C(t)$  is of rank  $n$  and  $D(t) = 0$ . In this special case, an  $n^2$ -order solution for state estimation can be obtained from (91) and

$$\hat{x}(t | T) = C^\dagger(t)\hat{y}(t | T), \quad (98)$$

where

$$C^\dagger(t) = \left( C^T(t)C(t) \right)^{-1} C^T(t) \quad (99)$$

denotes the Moore-Penrose pseudoinverse.

#### 6.5.4.4 Performance

An analysis of minimum-variance smoother performance requires an identity which is described after introducing some additional notation. Let  $\alpha = \mathcal{G}_0 w$  denote the output of linear time-varying system having the realisation

$$\dot{x}(t) = A(t)x(t) + w(t) \quad (100)$$

$$\alpha(t) = x(t), \quad (101)$$

where  $w(t) \in \mathbb{R}^n$  and  $A(t) \in \mathbb{R}^{n \times n}$ . By inspection of (100) – (101), the output of the inverse system  $w = \mathcal{G}_0^{-1}y$  is given by

$$w(t) = \dot{\alpha}(t) - A(t)\alpha(t). \quad (102)$$

Similarly, let  $\beta = \mathcal{G}_0^H u$  denote the output of the adjoint system  $\mathcal{G}_0^H$ , which from Lemma 1 has the realisation

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“The mind likes a strange idea as little as the body likes a strange protein and resists it with similar energy. It would not perhaps be too fanciful to say that a new idea is the most quickly acting antigen known to science. If we watch ourselves honestly we shall often find that we have begun to argue against a new idea even before it has been completely stated.” *Wilfred Batten Lewis Trotter*

$$-\dot{\zeta}(t) = A^T(t)\zeta(t) + u(t) \quad (103)$$

$$\beta(t) = \zeta(t). \quad (104)$$

It follows that the output of the inverse system  $u = \mathcal{G}_0^{-H} \beta$  is given by

$$u(t) = -\dot{\beta}(t) - A^T(t)\beta(t). \quad (105)$$

The following identity is required in the characterisation of smoother performance

$$-P(t)A^T(t) - A(t)P(t) = P(t)\mathcal{G}_0^{-H} + \mathcal{G}_0^{-1}P(t), \quad (106)$$

where  $P(t)$  is an arbitrary matrix of compatible dimensions. The above equation can be verified by using (102) and (105) within (106). Using the above notation, the exact Wiener-Hopf factor satisfies

$$\Delta\Delta^H = C(t)\mathcal{G}_0 B(t)Q(t)B^T(t)\mathcal{G}_0^H C^T(t) + R(t). \quad (107)$$

It is observed below that the approximate Wiener-Hopf factor (86) approaches the exact Wiener Hopf-factor whenever the problem is locally stationary, that is, whenever  $A(t)$ ,  $B(t)$ ,  $C(t)$ ,  $Q(t)$  and  $R(t)$  change sufficiently slowly, so that  $\dot{P}(t)$  of (87) approaches the zero matrix.

**Lemma 10 [8]:** *In respect of the signal model (1) – (2) with  $D(t) = 0$ ,  $E\{w(t)\} = E\{v(t)\} = 0$ ,  $E\{w(t)w^T(t)\} = Q(t)$ ,  $E\{v(t)v^T(t)\} = R(t)$ ,  $E\{w(t)v^T(t)\} = 0$  and the quantities defined above,*

$$\hat{\Delta}\hat{\Delta}^H = \Delta\Delta^H - C(t)\mathcal{G}_0 \dot{P}(t)\mathcal{G}_0^H C^T(t). \quad (108)$$

**Proof:** *The approximate Wiener-Hopf factor may be written as  $\hat{\Delta} = C(t)\mathcal{G}_0 K(t)R^{1/2}(t) + R^{1/2}(t)$ . It is easily shown that  $\hat{\Delta}\hat{\Delta}^H = C(t)\mathcal{G}_0 (P\mathcal{G}_0^{-H} + \mathcal{G}_0^{-1}P + K(t)R(t)K^T(t))\mathcal{G}_0^H C^T(t)$  and using (106) gives  $\hat{\Delta}\hat{\Delta}^H = C(t)\mathcal{G}_0 (B(t)Q(t)B^T(t) - \dot{P}(t)\mathcal{G}_0^H C^T(t) + R(t))$ . The result follows by comparing  $\hat{\Delta}\hat{\Delta}^H$  and (107).*

Consequently, the minimum-variance smoother (88) achieves the best-possible estimator performance, namely  $\|\mathcal{R}_{ei2}\mathcal{R}_{ei2}^H\|_2 = 0$ , whenever the problem is locally stationary.

**Lemma 11 [8]:** *The output estimation smoother (88) satisfies*

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“Every great advance in natural knowledge has involved the absolute rejection of authority.” *Thomas Henry Huxley*

$$\mathcal{R}_{ei2} = R(t)[(\Delta\Delta^H)^{-1} - (\Delta\Delta^H - C(t)\mathcal{G}_0\dot{P}(t)\mathcal{G}_0^H C^T(t))^{-1}]\Delta. \quad (109)$$

**Proof:** Substituting (88) into (77) yields

$$\mathcal{R}_{ei2} = R(t)[(\Delta\Delta^H)^{-1} - (\hat{\Delta}\hat{\Delta}^H)^{-1}]\Delta. \quad (110)$$

The result is now immediate from (108) and (110).

Conditions for the convergence of the Riccati difference equation solution (87) and hence the asymptotic optimality of the smoother (88) are set out below.

**Lemma 12 [8]:** Let  $S(t) = C^T(t)R^{-1}(t)C(t)$ . If (i) there exist solutions  $P(t) \geq P(t+\delta_t)$  of (87) for a  $t > \delta_t > 0$ ; and

(ii)

$$\begin{bmatrix} Q(t) & A(t) \\ A^T(t) & -S(t) \end{bmatrix} \geq \begin{bmatrix} Q(t+\delta_t) & A(t+\delta_t) \\ A^T(t+\delta_t) & -S(t+\delta_t) \end{bmatrix} \quad (111)$$

for all  $t > \delta_t$  then

$$\lim_{t \rightarrow \infty} \|\mathcal{R}_{yi2} \mathcal{R}_{yi2}^H\|_2 = 0. \quad (112)$$

**Proof:** Conditions (i) and (ii) together with Theorem 1 imply  $P(t) \geq P(t+\delta_t)$  for all  $t > \delta_t$  and  $\lim_{t \rightarrow \infty} \dot{P}(t) = 0$ . The claim (112) is now immediate from Lemma 11.

### 6.5.5 Performance Comparison

The following scalar time-invariant examples compare the performance of the minimum-variance filter (92), maximum-likelihood smoother (50), Fraser-Potter smoother (73) and minimum-variance smoother (88) under Gaussian and nongaussian noise conditions.

**Example 5 [9].** Suppose that  $A = -1$  and  $B = C = Q = 1$ . Simulations were conducted using  $T = 100$  s,  $\delta t = 1$  ms and 1000 realisations of Gaussian noise processes. The mean-square-error (MSE) exhibited by the filter and smoothers as a function of the input signal-to-noise ratio (SNR) is shown in Fig. 2. As expected, it can be seen that the smoothers outperform the filter. Although the minimum-variance smoother exhibits the lowest mean-square error, the performance benefit diminishes at high signal-to-noise ratios.

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“The definition of insanity is doing the same thing over and over again and expecting different results.”  
Albert Einstein

**Example 6 [9].** Suppose instead that the process noise is the unity-variance deterministic signal  $w(t) = \sin(t)\sigma_{\sin(t)}^{-1}$ , where  $\sigma_{\sin(t)}^2$  denotes the sample variance of  $\sin(t)$ . The results of simulations employing the sinusoidal process noise and Gaussian measurement noise are shown in Fig. 3. Once again, the smoothers exhibit better performance than the filter. It can be seen that the minimum-variance smoother provides the best mean-square-error performance. The minimum-variance smoother appears to be less perturbed by nongaussian noises because it does not rely on assumptions about the underlying distributions.

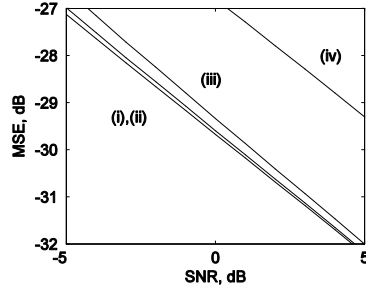


Fig. 2. MSE versus SNR for Example 4: (i) minimum-variance smoother, (ii) Fraser-Potter smoother, (iii) maximum-likelihood smoother and (iv) minimum-variance filter.

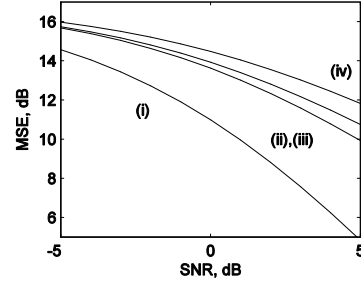


Fig. 3. MSE versus SNR for Example 5: (i) minimum-variance smoother, (ii) Fraser-Potter smoother, (iii) maximum-likelihood smoother and (iv) minimum-variance filter.

## 6.6 Chapter Summary

The fixed-point smoother produces state estimates at some previous point in time, that is,

$$\hat{\xi}(t) = \Sigma(t)C^T(t)R^{-1}(t)(z(t) - C(t)\hat{x}(t|t)),$$

where  $\Sigma(t)$  is the smoother error covariance.

In fixed-lag smoothing, state estimates are calculated at a fixed time delay  $\tau$  behind the current measurements. This smoother has the form

$$\begin{aligned} \hat{x}(t|t+\tau) &= A(t)\hat{x}(t|t+\tau) + B(t)Q(t)B^T(t)P^{-1}(t)(\hat{x}(t|t+\tau) - \hat{x}(t|t)) \\ &\quad + P(t)\Phi^T(t+\tau, t)C^T(t+\tau)R^{-1}(t+\tau)(z(t+\tau) - C(t+\tau)\hat{x}(t+\tau)), \end{aligned}$$

where  $\Phi(t+\tau, t)$  is the transition matrix of the minimum-variance filter.

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“The first problem for all of us, men and women, is not to learn but to unlearn.” *Gloria Marie Steinem*

Three common fixed-interval smoothers are listed in Table 1, which are for retrospective (or off-line) data analysis. The Rauch-Tung-Streibel (RTS) smoother and Fraser-Potter (FP) smoother are minimum-order solutions. The RTS smoother differential equation evolves backward in time, in which  $G(\tau) = B(\tau)Q(\tau)B^T(\tau)P^{-1}(\tau)$  is the smoothing gain. The FP smoother employs a linear combination of forward state estimates and backward state estimates obtained by running a filter over the time-reversed measurements. The optimum minimum-variance solution, in which  $\bar{A}(t) = A(t) - K(t)C(t)$ , in which  $K(t)$  is the predictor gain, involves a cascade of forward and adjoint predictions. It can be seen that the optimum minimum-variance smoother is the most complex and so any performance benefits need to be reconciled with the increased calculation cost.

	ASSUMPTIONS	MAIN RESULTS
Signals and system	$E\{w(t)\} = E\{v(t)\} = 0$ . $E\{w(t)w^T(t)\} = Q(t) > 0$ and $E\{v(t)v^T(t)\} = R(t) > 0$ are known. $A(t)$ , $B(t)$ and $C(t)$ are known.	$\dot{x}(t) = A(t)x(t) + B(t)w(t)$ $y(t) = C(t)x(t)$ $z(t) = y(t) + v(t)$
RTS smoother	Assumes that the filtered and smoothed states are normally distributed. $\hat{x}(t t)$ previously calculated by Kalman filter.	$-\dot{\hat{x}}(\tau T) = A(\tau)\hat{x}(\tau T) + G(\tau)(\hat{x}(\tau T) - \hat{x}(\tau \tau))$
FP smoother	$\hat{x}(t t)$ previously calculated by Kalman filter.	$\hat{x}(t T) = (P^{-1}(t t) + \Sigma^{-1}(t t))^{-1}$ $\times (P^{-1}(t t)\hat{x}(t t) + \Sigma^{-1}(t t)\xi(t t))$
Optimal smoother		$\begin{bmatrix} \dot{\hat{x}}(t) \\ \alpha(t) \end{bmatrix} = \begin{bmatrix} \bar{A}(t) & K(t) \\ -R^{-1/2}(t)C(t) & R^{-1/2}(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ z(t) \end{bmatrix}$ $\begin{bmatrix} -\dot{\xi}(t) \\ \beta(t) \end{bmatrix} = \begin{bmatrix} \bar{A}^T(t) & -C^T(t)R^{-1/2}(t) \\ K^T(t) & R^{-1/2}(t) \end{bmatrix} \begin{bmatrix} \xi(t) \\ \alpha(t) \end{bmatrix}$ $\hat{y}(t T) = z(t) - R(t)\beta(t)$

Table 1. Continuous-time fixed-interval smoothers.

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“Remember a dead fish can float downstream but it takes a live one to swim upstream.” *William Claude Fields*

The output estimation error covariance for the general estimation problem can be written as  $\mathcal{R}_{ei}\mathcal{R}_{ei}^H = \mathcal{R}_{ei1}\mathcal{R}_{ei1}^H + \mathcal{R}_{ei2}\mathcal{R}_{ei2}^H$ , where  $\mathcal{R}_{ei1}\mathcal{R}_{ei1}^H$  specifies a lower performance bound and  $\mathcal{R}_{ei2}\mathcal{R}_{ei2}^H$  is a function of the estimator solution. The optimal smoother solution achieves  $\|\{\mathcal{R}_{ei2}\mathcal{R}_{ei2}^H\}_+\|_2 = 0$  and provides the best mean-square-error performance, provided of course that the problem assumptions are correct. The minimum-variance smoother solution also attains best-possible performance whenever the problem is locally stationary, that is, when  $A(t)$ ,  $B(t)$ ,  $C(t)$ ,  $Q(t)$  and  $R(t)$  change sufficiently slowly.

## 6.7 Problems

**Problem 1.** Write down augmented state-space matrices  $A^{(a)}(t)$ ,  $B^{(a)}(t)$  and  $C^{(a)}(t)$  for the continuous-time fixed-point smoother problem.

- (i) Substitute the above matrices into  $\dot{P}^{(a)}(t) = A^{(a)}(t)P^{(a)}(t) + P^{(a)}(t)(A^{(a)})^T(t) - P^{(a)}(t)(C^{(a)})^T(t)R^{-1}(t)C^{(a)}(t)P^{(a)}(t) + B^{(a)}(t)Q(t)(B^{(a)}(t))^T$  to obtain the component Riccati differential equations.
- (ii) Develop expressions for the continuous-time fixed-point smoother estimate and the smoother gain.

**Problem 2.** The Hamiltonian equations (60) were derived from the forward version of the maximum likelihood smoother (54). Derive the alternative form

$$\begin{bmatrix} -\dot{\hat{x}}(t|T) \\ \dot{\xi}(t|T) \end{bmatrix} = \begin{bmatrix} A(t) & B(t)Q(t)B^T(t) \\ C^T(t)R^{-1}(t)C(t) & A^T(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t|T) \\ \xi(t|T) \end{bmatrix} - \begin{bmatrix} 0 \\ C^T(t)R^{-1}(t)z(t) \end{bmatrix}.$$

from the backward smoother (50). Hint: use the backward Kalman-Bucy filter and the backward Riccati equation.

**Problem 3.** It is shown in [6] and [17] that the intermediate variable within the Hamiltonian equations (60) is given by

$$\xi(t|T) = \int_t^T \Phi^T(s,t)C^T(s)R^{-1}(s)(z(s) - C(s)\hat{x}(s|s))ds,$$

where  $\Phi^T(s,t)$  is the transition matrix of the Kalman-Bucy filter. Use the above equation to derive

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“He who rejects change is the architect of decay. The only human institution which rejects progress is the cemetery.” *James Harold Wilson*



$$-\dot{\xi}(t|T) = -C^T(t)R^{-1}(t)C(t)\hat{x}(t|T) + A^T(t)\xi(t) - C^T(t)R^{-1}(t)z(t).$$

**Problem 4.** Show that the adjoint of system having state space parameters  $\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$  is parameterised by  $\begin{bmatrix} -A^T(t) & -C^T(t) \\ B^T(t) & D^T(t) \end{bmatrix}$ .

**Problem 5.** Suppose  $\mathcal{G}_0$  is a system parameterised by  $\begin{bmatrix} A(t) & I \\ I & 0 \end{bmatrix}$ , show that  $-P(t)A^T(t) - A(t)P(t) = P(t)\mathcal{G}_0^{-H} + \mathcal{G}_0^{-1}P(t)$ .

**Problem 6.** The optimum minimum-variance smoother was developed by finding the solution that minimises  $\|\tilde{y}_2\tilde{y}_2^H\|_2$ . Use the same completing-the-square approach to find the optimum minimum-variance filter. (Hint: Find the solution that minimises  $\|\tilde{y}_2\tilde{y}_2^T\|_2$ .)

**Problem 7 [9].** Derive the output estimation minimum-variance filter by finding a solution. Let  $a \in \mathbb{R}$ ,  $b = I$ ,  $c \in \mathbb{R}$  and  $d = 0$  denote the time-invariant state-space parameters of the plant  $\mathcal{G}$ . Denote the error covariance, gain of the Kalman filter and gain of the maximum-likelihood smoother by  $p$ ,  $k$  and  $g$ , respectively. Show that

$$\begin{aligned} H_1(s) &= k(s-a+kc)^{-1}, \\ H_2(s) &= c g k (-s-a+g)^{-1} (s-a+kc)^{-1}, \\ H_3(s) &= k c (-a+kc) (s-a+kc)^{-1} (-s-a+kc)^{-1}, \\ H_4(s) &= ((-a+kc)^2 - (-a+kc-k)^2) (s-a+kc)^{-1} (-s-a+kc)^{-1} \end{aligned}$$

are the transfer functions of the Kalman filter, maximum-likelihood smoother, the Fraser-Potter smoother and the minimum variance smoother,<sup>32</sup> respectively.

**Problem 8.**

- (i) Develop a state-space formulation of an approximate Wiener-Hopf factor for the case when the plant includes a nonzero direct feedthrough matrix (that is,  $D(t) \neq 0$ ).
- (ii) Use the matrix inversion lemma to obtain the inverse of the approximate Wiener-Hopf factor for the minimum-variance smoother.

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“It is not the strongest of the species that survive, nor the most intelligent, but the most responsive to change.” *Charles Robert Darwin*

## 6.8 Glossary

$p(x(t))$	Probability density function of a continuous random variable $x(t)$ .
$x(t) \sim \mathcal{N}(\mu, R_{xx})$	The random variable $x(t)$ has a normal distribution with mean $\mu$ and covariance $R_{xx}$ .
$f(x(t))$	Cumulative distribution function or likelihood function of $x(t)$ .
$\hat{x}(t   t + \tau)$	Estimate of $x(t)$ at time $t$ given data at fixed time lag $\tau$ .
$\hat{x}(t   T)$	Estimate of $x(t)$ at time $t$ given data over a fixed interval $T$ .
$\hat{w}(t   T)$	Estimate of $w(t)$ at time $t$ given data over a fixed interval $T$ .
$G(t)$	Gain of the minimum-variance smoother developed by Rauch, Tung and Striebel.
$\mathcal{R}_{ei}$	A linear system that operates on the inputs $i = \begin{bmatrix} v^T & w^T \end{bmatrix}^T$ and generates the output estimation error $e$ .
$C^\#(t)$	Moore-Penrose pseudoinverse of $C(t)$ .
$\Delta$	The Wiener-Hopf factor which satisfies $\Delta\Delta^H = \mathcal{G}Q\mathcal{G}^H + R$ .
$\hat{\Delta}$	Approximate Wiener-Hopf factor.
$\hat{\Delta}^H$	The adjoint of $\hat{\Delta}$ .
$\hat{\Delta}^{-H}$	The inverse of $\hat{\Delta}^H$ .
$\{\hat{\Delta}^{-H}\}_+$	The causal part of $\hat{\Delta}^{-H}$ .

## 6.9 References

- [1] J. S. Meditch, "A Survey of Data Smoothing for Linear and Nonlinear Dynamic Systems", *Automatica*, vol. 9, pp. 151-162, 1973
- [2] T. Kailath, "A View of Three Decades of Linear Filtering Theory", *IEEE Transactions on Information Theory*, vol. 20, no. 2, pp. 146 – 181, Mar., 1974.
- [3] H. E. Rauch, F. Tung and C. T. Striebel, "Maximum Likelihood Estimates of Linear Dynamic Systems", *AIAA Journal*, vol. 3, no. 8, pp. 1445 – 1450, Aug., 1965.
- [4] D. C. Fraser and J. E. Potter, "The Optimum Linear Smoother as a Combination of Two Optimum Linear Filters", *IEEE Transactions on Automatic Control*, vol. AC-14, no. 4, pp. 387 – 390, Aug., 1969.
- [5] J. S. Meditch, "On Optimal Fixed Point Linear Smoothing", *International*

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"Once a new technology rolls over you, if you're not part of the steamroller, you're part of the road."  
Stewart Brand

- Journal of Control*, vol. 6, no. 2, pp. 189 – 199, 1967.
- [6] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York, 1971.
  - [7] J. B. Moore, “Fixed-Lag Smoothing Results for Linear Dynamical Systems”, *A.T.R.*, vol. 7, no. 2, pp. 16 – 21, 1973.
  - [8] G. A. Einicke, “Asymptotic Optimality of the Minimum-Variance Fixed-Interval Smoother”, *IEEE Transactions on Signal Processing*, vol. 55, no. 4, pp. 1543 – 1547, Apr. 2007.
  - [9] G. A. Einicke, J. C. Ralston, C. O. Hargrave, D. C. Reid and D. W. Hainsworth, “Longwall Mining Automation, An Application of Minimum-Variance Smoothing”, *IEEE Control Systems Magazine*, vol. 28, no. 6, pp. 28 – 37, Dec. 2008.
  - [10] G. A. Einicke, “A Solution to the Continuous-Time H-infinity Fixed-Interval Smoother Problem”, *IEEE Transactions on Automatic Control*, vol. 54, no. 12, pp. 2904 – 2908, Dec. 2009.
  - [11] D. J. N. Limebeer, B. D. O. Anderson, P. Khargonekar and M. Green, “A Game Theoretic Approach to  $H_\infty$  Control for Time-varying Systems”, *SIAM Journal on Control and Optimization*, vol. 30, no. 2, pp. 262 – 283, 1992.
  - [12] G. Freiling, G. Jank and H. Abou-Kandil, “Generalized Riccati Difference and Differential Equations”, *Linear Algebra And Its Applications*, pp. 291 – 303, 199
  - [13] L. E. Zachrisson, “On Optimal Smoothing of Continuous Time Kalman Processes”, *Information Sciences*, vol. 1, pp. 143 – 172, 1969.
  - [14] U. Shaked and Y. Theodore, “ $H_\infty$  – Optimal Estimation: A Tutorial”, *Proc. 31<sup>st</sup> IEEE Conference on Decision Control*, Ucsn, Arizona, pp. 2278 – 2286, Dec. 1992.
  - [15] J. S. Meditch, “On Optimal Linear Smoothing Theory”, *Information and Control*, vol. 10, pp. 598 – 615, 1967.
  - [16] P. S. Maybeck, *Stochastic Models, Estimation and Control*, Vol. 2, Academic press, New York, 1982.
  - [17] T. Kailath, A. H. Sayed and B. Hassibi, *Linear Estimation*, Pearson Prentice Hall, New Jersey, 2000.
  - [18] B. D. O. Anderson, “Properties of Optimal Linear Smoothing”, *IEEE Transactions on Automatic Control*, vol. 14, no. 1, pp. 114 – 115, Feb. 1969.
  - [19] S. Chirarattananon and B. D. O. Anderson, “Stable Fixed-Lag Smoothing of Continuous-time Processes”, *IEEE Transactions on Information Theory*, vol. 20, no. 1, pp. 25 – 36, Jan. 1974.
  - [20] A. Gelb, *Applied Optimal Estimation*, The Analytic Sciences Corporation, USA, 1974.
  - [21] L. Ljung and T. Kailath, “A Unified Approach to Smoothing Formulas”, *Automatica*, vol. 12, pp. 147 – 157, 197
  - [22] T. Kailath and P. Frost, “An Innovations Approach to Least-Squares Estimation Part II: Linear Smoothing in Additive White Noise”, *IEEE Transactions on Automat. Control*, vol. 13, no. 6, pp. 655 – 660, 1968.

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“In times of profound change, the learners inherit the earth, while the learned find themselves beautifully equipped to deal with a world that no longer exists.” *Al Rogers*

- [23] F. L. Lewis, L. Xie and D. Popa, *Optimal and Robust Estimation With an Introduction to Stochastic Control Theory*, Second Edition, CRC Press, Taylor & Francis Group, 2008.
- [24] F. A. Badawi, A. Lindquist and M. Pavon, "A Stochastic Realization Approach to the Smoothing Problem", *IEEE Transactions on Automatic Control*, vol. 24, no. 6, Dec. 1979.
- [25] J. A. Rice, *Mathematical Statistics and Data Analysis* (Second Edition), Duxbury Press, Belmont, California, 1995.
- [26] R. G. Brown and P. Y. C. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, Second Edition, John Wiley & Sons, Inc., New York, 1992.
- [27] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1995.

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"I don't want to be left behind. In fact, I want to be here before the action starts." *Kerry Francis Bullmore Packer*

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## 7. Discrete-Time Minimum-Variance Smoothing

### 7.1 Introduction

Observations are invariably accompanied by measurement noise and optimal filters are the usual solution of choice. Filter performances that fall short of user expectations motivate the pursuit of smoother solutions. Smoothers promise useful mean-square-error improvement at mid-range signal-to-noise ratios, provided that the assumed model parameters and noise statistics are correct.

In general, discrete-time filters and smoothers are more practical than the continuous-time counterparts. Often a designer may be able to value-add by assuming low-order discrete-time models which bear little or no resemblance to the underlying processes. Continuous-time approaches may be warranted only when application-specific performance considerations outweigh the higher overheads.

This chapter canvasses the main discrete-time fixed-point, fixed-lag and fixed interval smoothing results [1] – [9]. Fixed-point smoothers [1] calculate an improved estimate at a prescribed past instant in time. Fixed-lag smoothers [2] – [3] find application where small end-to-end delays are tolerable, for example, in press-to-talk communications or receiving public broadcasts. Fixed-interval smoothers [4] – [9] dispense with the need to fine tune the time of interest or the smoothing lags. They are suited to applications where processes are staggered such as delayed control or off-line data analysis. For example, in underground coal mining, smoothed position estimates and control signals can be calculated while a longwall shearer is momentarily stationary at each end of the face [9]. Similarly, in exploration drilling, analyses are typically carried out post-data acquisition.

The smoother descriptions are organised as follows. Section 7.2 sets out two prerequisites: time-varying adjoint systems and Riccati difference equation comparison theorems. Fixed-point, fixed-lag and fixed-interval smoothers are

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“An inventor is simply a person who doesn't take his education too seriously. You see, from the time a person is six years old until he graduates from college he has to take three or four examinations a year. If he flunks once, he is out. But an inventor is almost always failing. He tries and fails maybe a thousand times. If he succeeds once then he's in. These two things are diametrically opposite. We often say that the biggest job we have is to teach a newly hired employee how to fail intelligently. We have to train him to experiment over and over and to keep on trying and failing until he learns what will work.” *Charles Franklin Kettering*

discussed in Sections 7.3, 7.4 and 7.5, respectively. It turns out that the structures of the discrete-time smoothers are essentially the same as those of the previously-described continuous-time versions. Differences arise in the calculation of Riccati equation solutions and the gain matrices. Consequently, the treatment is somewhat condensed. It is reaffirmed that the above-mentioned smoothers outperform the Kalman filter and the minimum-variance smoother provides the best performance.

## 7.2 Prerequisites

### 7.2.1 Time-varying Adjoint Systems

Consider a linear time-varying system,  $\mathcal{G}$ , operating on an input,  $w$ , namely,  $y = \mathcal{G} w$ . Here,  $w = [w_1, \dots, w_N] \in \mathbb{R}^{m \times N}$ , denotes the matrix of  $w_k \in \mathbb{R}^m$  over an interval  $k \in [1, N]$ . It is assumed that  $\mathcal{G}: \mathbb{R}^{m \times N} \rightarrow \mathbb{R}^{p \times N}$  has the state-space realisation

$$x_{k+1} = A_k x_k + B_k w_k, \quad (1)$$

$$y_k = C_k x_k + D_k w_k. \quad (2)$$

As before, the adjoint system,  $\mathcal{G}^H$ , satisfies

$$\langle y, \mathcal{G} w \rangle = \langle \mathcal{G}^H y, w \rangle \quad (3)$$

for all  $y \in \mathbb{R}^{p \times N}$  and  $w \in \mathbb{R}^{m \times N}$ .

**Lemma 1:** In respect of the system  $\mathcal{G}$  described by (1) – (2), with  $x_0 = 0$ , the adjoint system  $\mathcal{G}^H$  having the realisation

$$\zeta_{k-1} = A_k^T \zeta_k - C_k^T u_k, \quad (4)$$

$$z_k = -B_k^T \zeta_k + D_k^T u_k, \quad (5)$$

with  $\zeta_N = 0$ , satisfies (3).

A proof appears in [7] and proceeds similarly to that within Lemma 1 of Chapter 2. The simplification  $D_k = 0$  is assumed below unless stated otherwise.

### 7.2.2 Riccati Equation Comparison

The ensuing performance comparisons of filters and smoothers require methods for comparing the solutions of Riccati difference equations which are developed

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“If you’re not failing every now and again, it’s a sign you’re not doing anything very innovative.”  
(Woody) Allen Stewart Konigsberg

below. Simplified Riccati difference equations which do not involve the  $B_k$  and measurement noise covariance matrices are considered initially. A change of variables for the more general case is stated subsequently.

Suppose there exist  $A_{t+k-1} \in \mathbb{R}^{n \times n}$ ,  $\tilde{C}_{t+k-1} \in \mathbb{R}^{p \times n}$ ,  $\tilde{Q}_{t+k-1} = \tilde{Q}_{t+k-1}^T \in \mathbb{R}^{n \times n}$  and  $P_{t+k-1} = P_{t+k-1}^T \in \mathbb{R}^{n \times n}$  for a  $t \geq 0$  and  $k \geq 0$ . Following the approach of Wimmer [10], define the Riccati operator

$$\Phi(P_{t+k-1}, A_{t+k-1}, \tilde{C}_{t+k-1}, \tilde{Q}_{t+k-1}) = A_{t+k-1} P_{t+k-1} A_{t+k-1}^T + \tilde{Q}_{t+k-1} - A_{t+k-1} P_{t+k-1} \tilde{C}_{t+k-1}^T (I + \tilde{C}_{t+k-1} P_{t+k-1} \tilde{C}_{t+k-1}^T)^{-1} \tilde{C}_{t+k-1} P_{t+k-1} A_{t+k-1}^T. \quad (6)$$

Let  $\Gamma_{t+k-1} = \begin{bmatrix} A_{t+k-1} & -\tilde{C}_{t+k-1}^T \tilde{C}_{t+k-1} \\ -\tilde{Q}_{t+k-1} & -A_{t+k-1}^T \end{bmatrix}$  denote the Hamiltonian matrix

corresponding to  $\Phi(P_{t+k-1}, A_{t+k-1}, \tilde{C}_{t+k-1}, \tilde{Q}_{t+k-1})$  and define  $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ , in which

$I$  is an identity matrix of appropriate dimensions. It is known that solutions of (6) are monotonically dependent on  $J\Gamma_{k+1} = \begin{bmatrix} \tilde{Q}_{t+k-1} & A_{t+k-1}^T \\ A_{t+k-1} & -\tilde{C}_{t+k-1}^T \tilde{C}_{t+k-1} \end{bmatrix}$ . Consider a

second Riccati operator employing the same initial solution  $P_{t+k-1}$  but different state-space parameters

$$\Phi(P_{t+k-1}, A_{t+k}, \tilde{C}_{t+k}, \tilde{Q}_{t+k}) = A_{t+k} P_{t+k-1} A_{t+k}^T + \tilde{Q}_{t+k} - A_{t+k} P_{t+k-1} \tilde{C}_{t+k}^T (I + \tilde{C}_{t+k} P_{t+k-1} \tilde{C}_{t+k}^T)^{-1} \tilde{C}_{t+k} P_{t+k-1} A_{t+k}^T. \quad (7)$$

The following theorem, which is due to Wimmer [10], compares the above two Riccati operators.

**Theorem 1: [10]:** Suppose that

$$\begin{bmatrix} \tilde{Q}_{t+k-1} & A_{t+k-1}^T \\ A_{t+k-1} & -\tilde{C}_{t+k-1}^T \tilde{C}_{t+k-1} \end{bmatrix} \geq \begin{bmatrix} \tilde{Q}_{t+k} & A_{t+k}^T \\ A_{t+k} & -\tilde{C}_{t+k}^T \tilde{C}_{t+k} \end{bmatrix}$$

for a  $t \geq 0$  and for all  $k \geq 0$ . Then

$$\Phi(P_{t+k-1}, A_{t+k-1}, \tilde{C}_{t+k-1}, \tilde{Q}_{t+k-1}) \geq \Phi(P_{t+k-1}, A_{t+k}, \tilde{C}_{t+k}, \tilde{Q}_{t+k}) \quad (8)$$

for all  $k \geq 0$ .

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“Although personally I am quite content with existing explosives, I feel we must not stand in the path of improvement.” *Winston Leonard Spencer-Churchill*



The above result underpins the following more general Riccati difference equation comparison theorem.

**Theorem 2:** [11], [8]: *With the above definitions, suppose for a  $t \geq 0$  and for all  $k \geq 0$  that:*

- (i) *there exists a  $P_t \geq P_{t+1}$  and*
- (ii) 
$$\begin{bmatrix} \tilde{Q}_{t+k-1} & A_{t+k-1}^T \\ A_{t+k-1} & -\tilde{C}_{t+k-1}^T \tilde{C}_{t+k-1} \end{bmatrix} \geq \begin{bmatrix} \tilde{Q}_{t+k} & A_{t+k}^T \\ A_{t+k} & -\tilde{C}_{t+k}^T \tilde{C}_{t+k} \end{bmatrix}.$$

*Then  $P_{t+k} \geq P_{t+k+1}$  for all  $k \geq 0$ .*

**Proof:** Assumption (i) is the  $k = 0$  case for an induction argument. For the inductive step, denote  $P_{t+k} = \Phi(P_{t+k-1}, A_{t+k-1}, \tilde{C}_{t+k-1}, \tilde{Q}_{t+k-1})$  and  $P_{t+k+1} = \Phi(P_{t+k}, A_{t+k}, \tilde{C}_{t+k}, \tilde{Q}_{t+k})$ . Then

$$\begin{aligned} P_{t+k} - P_{t+k+1} &= (\Phi(P_{t+k-1}, A_{t+k-1}, \tilde{C}_{t+k-1}, \tilde{Q}_{t+k-1}) - \Phi(P_{t+k-1}, A_{t+k}, \tilde{C}_{t+k}, \tilde{Q}_{t+k})) \\ &\quad + (\Phi(P_{t+k-1}, A_{t+k}, \tilde{C}_{t+k}, \tilde{Q}_{t+k}) - \Phi(P_{t+k}, A_{t+k}, \tilde{C}_{t+k}, \tilde{Q}_{t+k})). \end{aligned} \quad (9)$$

The first term on the right-hand-side of (9) is non-negative by virtue of Assumption (ii) and Theorem 1. By appealing to Theorem 2 of Chapter 5, the second term on the right-hand-side of (9) is non-negative and thus  $P_{t+k} - P_{t+k+1} \geq 0$ .

A change of variables [8]  $\tilde{C}_k = R_k^{-1/2} C_k$  and  $\tilde{Q}_k = B_k Q_k B_k^T$ , allows the application of Theorem 2 to the more general forms of Riccati differential equations.

## 7.3 Fixed-Point Smoothing

### 7.3.1 Solution Derivation

The development of a discrete-time fixed-point smoother follows the continuous-time case. An innovation by Zachrisson [12] involves transforming the smoothing problem into a filtering problem that possesses an augmented state. Following the approach in [1], consider an augmented state vector  $x_k^{(a)} = \begin{bmatrix} x_k \\ \xi_k \end{bmatrix}$  for the signal model

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“Never before in history has innovation offered promise of so much to so many in so short a time.”  
William Henry (Bill) Gates III

$$x_{k+1}^{(a)} = A_k^{(a)} x_k^{(a)} + B_k^{(a)} w_k, \quad (10)$$

$$z_k = C_k^{(a)} x_k^{(a)} + v_k, \quad (11)$$

where  $A_k^{(a)} = \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix}$ ,  $B_k^{(a)} = \begin{bmatrix} B_k \\ 0 \end{bmatrix}$  and  $C_k^{(a)} = [C_k \ 0]$ . It can be seen that the first component of  $x_k^{(a)}$  is  $x_k$ , the state of the system  $x_{k+1} = A_k x_k + B_k w_k$ ,  $y_k = C_k x_k + v_k$ . The second component,  $\xi_k$ , equals  $x_k$  at time  $k = \tau$ , that is,  $\xi_k = x_\tau$ . The objective is to calculate an estimate  $\hat{\xi}_k$  of  $\xi_k$  at time  $k = \tau$  from measurements  $z_k$  over  $k \in [0, N]$ . A solution that minimises the variance of the estimation error is obtained by employing the standard Kalman filter recursions for the signal model (10) – (11). The predicted and corrected states are respectively obtained from

$$\hat{x}_{k/k}^{(a)} = (I - L_k^{(a)} C_k^{(a)}) \hat{x}_{k/k-1}^{(a)} + L_k^{(a)} z_k, \quad (12)$$

$$\hat{x}_{k+1/k}^{(a)} = A_k^{(a)} \hat{x}_{k/k}^{(a)} \quad (13)$$

$$= (A_k^{(a)} - K_k^{(a)} C_k^{(a)}) \hat{x}_{k/k-1}^{(a)} + K_k^{(a)} z_k, \quad (14)$$

where  $K_k = A_k^{(a)} L_k^{(a)}$  is the predictor gain,  $L_k^{(a)} = P_{k/k-1}^{(a)} (C_k^{(a)})^T (C_k^{(a)} P_{k/k-1}^{(a)} (C_k^{(a)})^T + R_k)^{-1}$  is the filter gain,

$$P_{k/k}^{(a)} = P_{k/k-1}^{(a)} - P_{k/k-1}^{(a)} (C_k^{(a)})^T (L_k^{(a)})^T \quad (15)$$

is the corrected error covariance and

$$P_{k+1/k}^{(a)} = A_k^{(a)} P_{k/k}^{(a)} (A_k^{(a)})^T + B_k^{(a)} Q_k (B_k^{(a)})^T \quad (16)$$

$$\begin{aligned} &= A_k^{(a)} P_{k/k-1}^{(a)} (A_k^{(a)})^T - A_k^{(a)} P_{k/k-1}^{(a)} (C_k^{(a)})^T (K_k^{(a)})^T + B_k^{(a)} Q_k (B_k^{(a)})^T \\ &= A_k^{(a)} P_{k/k-1}^{(a)} \left( (A_k^{(a)})^T - (C_k^{(a)})^T (K_k^{(a)})^T \right) + B_k^{(a)} Q_k (B_k^{(a)})^T \end{aligned} \quad (17)$$

is the predicted error covariance. The above Riccati difference equation is written in the partitioned form

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“You can’t just ask customers what they want and then try to give that to them. By the time you get it built, they’ll want something new.” *Steven Paul Jobs*

$$\begin{aligned}
P_{k+1/k}^{(a)} &= \begin{bmatrix} P_{k+1/k} & \Sigma_{k+1/k}^T \\ \Sigma_{k+1/k} & \Omega_{k+1/k} \end{bmatrix} \\
&= \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{k/k-1} & \Sigma_{k/k-1}^T \\ \Sigma_{k/k-1} & \Omega_{k/k-1} \end{bmatrix} \\
&\quad \times \left( \begin{bmatrix} A_k^T & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} C_k^T \\ 0 \end{bmatrix} \begin{bmatrix} K_k^T & \underline{K}_k^T \end{bmatrix} (C_k P_{k/k-1} C_k^T + R_k)^{-1} \right) \\
&\quad + \begin{bmatrix} B_k \\ 0 \end{bmatrix} Q_k \begin{bmatrix} B_k^T & 0 \end{bmatrix}, \tag{18}
\end{aligned}$$

in which the gains are given by

$$\begin{aligned}
K_k^{(a)} &= \begin{bmatrix} K_k \\ \underline{L}_k \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{k/k-1} & \Sigma_{k/k-1}^T \\ \Sigma_{k/k-1} & \Pi_{k/k-1} \end{bmatrix} \begin{bmatrix} C_k^T \\ 0 \end{bmatrix} (C_k P_{k/k-1} C_k^T + R_k)^{-1} \\
&= \begin{bmatrix} A_k P_{k/k-1} C_k^T \\ \Sigma_{k/k-1} C_k^T \end{bmatrix} (C_k P_{k/k-1} C_k^T + R_k)^{-1}, \tag{19}
\end{aligned}$$

see also [1]. The predicted error covariance components can be found from (18), viz.,

$$P_{k+1/k} = A_k P_{k/k} A_k^T + B_k Q_k B_k^T, \tag{20}$$

$$\Sigma_{k+1/k} = \Sigma_{k/k-1} (A_k^T - C_k^T K_k^T), \tag{21}$$

$$\Omega_{k+1/k} = \Omega_{k/k-1} - \Sigma_{k/k-1} C_k^T \underline{L}_k^T. \tag{22}$$

The sequences (21) – (22) can be initialised with  $\Sigma_{\tau+1/\tau} = P_{\tau/\tau}$  and  $\Omega_{\tau+1/\tau} = P_{\tau/\tau}$ . The state corrections are obtained from (12), namely,

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1}), \tag{23}$$

$$\hat{\xi}_{k/k} = \hat{\xi}_{k/k-1} + \underline{L}_k (z_k - C_k \hat{x}_{k/k-1}). \tag{24}$$

Similarly, the state predictions follow from (13),

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k}, \tag{25}$$

$$\hat{\xi}_{k+1/k} = \hat{\xi}_{k/k}. \tag{26}$$

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“Innovation is the whim of an elite before it becomes a need of the public.” *Ludwig von Mises*

In summary, the fixed-point smoother estimates for  $k \geq \tau$  are given by (24), which is initialised by  $\hat{\xi}_{\tau/\tau} = \hat{x}_{\tau/\tau}$ . The smoother gain is calculated as  $\underline{L}_k = \Sigma_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}$ , where  $\Sigma_{k/k-1}$  is given by (21).

### 7.3.2 Performance

It follows from the above that  $\Omega_{k+1/k} = \Omega_{k/k}$  and so

$$\Omega_{k+1/k+1} = \Omega_{k/k} - \Sigma_{k/k-1} C_k^T \underline{L}_k^T. \quad (27)$$

Next, it is argued that the discrete-time fixed-point smoother provides a performance improvement over the filter.

**Lemma 2 [1]:** *In respect of the fixed point smoother (24),*

$$P_{\tau/\tau} \geq \Omega_{k/k}. \quad (28)$$

**Proof:** *The recursion (22) may be written as the sum*

$$\Omega_{k+1/k+1} = \Omega_{\tau/\tau} - \sum_{i=\tau}^k \Sigma_{i/i-1} C_i^T (C_i P_{i/i-1} C_i^T + R)^{-1} C_i \Sigma_{i/i-1}, \quad (29)$$

where  $\Omega_{\tau/\tau} = P_{\tau/\tau}$ . Hence,  $P_{\tau/\tau} - \Omega_{k+1/k+1} = \sum_{i=\tau}^k \Sigma_{i/i-1} C_i^T (C_i P_{i/i-1} C_i^T + R)^{-1} C_i \Sigma_{i/i-1} \geq 0$ .

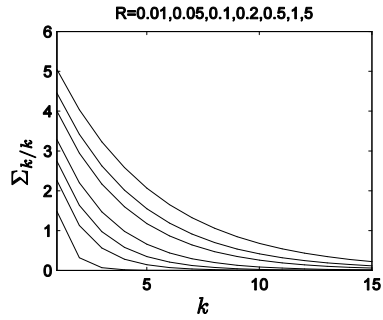


Fig. 1(a). Smoother estimation variance  $\Sigma_{k/k}$  versus  $k$  for Example 1.

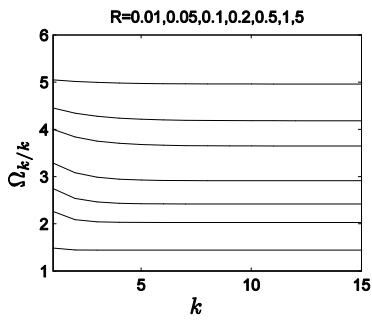


Fig. 1(b). Smoother estimation variances  $\Omega_{k+1/k+1}$  versus  $k$  for Example 1.

**Example 1.** Consider a first-order time-invariant plant, in which  $A = 0.9$ ,  $B = 1$ ,  $C = 0.1$  and  $Q = 1$ . An understanding of a fixed-point smoother's performance can be gleaned by examining the plots of the  $\Sigma_{k+1/k}$  and  $\Omega_{k+1/k+1}$  sequences shown in Fig. 1(a) and (b), respectively. The bottom lines of the figures correspond to measurement noise covariances of  $R = 0.01$  and the top lines correspond to  $R = 5$ . It can be seen for this example, that the  $\Sigma_{k/k}$  have diminishing impact after about 15 samples beyond the point of interest. From Fig. 1(b), it can be seen that smoothing appears most beneficial at mid-range measurement noise power, such as  $R = 0.2$ , since the plots of  $\Omega_{k+1/k+1}$  become flatter for  $R \geq 1$  and  $R \leq 0.05$ .

## 7.4 Fixed-Lag Smoothing

### 7.4.1 High-order Solution

Discrete-time fixed-lag smoothers calculate state estimates,  $\hat{x}_{k-N/k}$ , at time  $k$  given a delay of  $N$  steps. The objective is to minimise  $E\{(x_{k-N} - \hat{x}_{k-N/k})(x_{k-N} - \hat{x}_{k-N/k})^T\}$ . A common solution approach is to construct an augmented signal model that includes delayed states and then apply the standard Kalman filter recursions, see [1] – [3] and the references therein. Consider the signal model

$$\begin{bmatrix} x_{k+1} \\ x_k \\ x_{k-1} \\ \vdots \\ x_{k+1-N} \end{bmatrix} = \begin{bmatrix} A_k & 0 & \cdots & 0 \\ I_N & & & \\ 0 & I_N & & \\ \vdots & & \ddots & \\ 0 & 0 & & I_N & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \\ x_{k-N} \end{bmatrix} + \begin{bmatrix} B_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (30)$$

and

$$z_k = [C_k \quad 0 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \\ x_{k-N} \end{bmatrix} + v_k. \quad (31)$$

By applying the Kalman filter recursions to the above signal model, the predicted states are obtained as

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“You have to seek the simplest implementation of a problem solution in order to know when you’ve reached your limit in that regard. Then it’s easy to make tradeoffs, to back off a little, for performance reasons.” *Stephen Gary Wozniak*

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{x}_k \\ \hat{x}_{k-1} \\ \vdots \\ \hat{x}_{k+1-N} \end{bmatrix} = \begin{bmatrix} A_k & 0 & \cdots & 0 \\ I_N & & & 0 \\ 0 & I_N & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & & I_N & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{x}_{k-1} \\ \hat{x}_{k-2} \\ \vdots \\ \hat{x}_{k-N} \end{bmatrix} + \begin{bmatrix} K_{0,k} \\ K_{1,k} \\ K_{2,k} \\ \vdots \\ K_{N,k} \end{bmatrix} (z_k - C_k \hat{x}_{k/k-1}), \quad (32)$$

where  $K_{0,k}$ ,  $K_{1,k}$ ,  $K_{2,k}$ , ...,  $K_{N,k}$  denote the submatrices of the predictor gain. Two important observations follow from the above equation. First, the desired smoothed estimates  $\hat{x}_{k-1/k}$  ...  $\hat{x}_{k-N+1/k}$  are contained within the one-step-ahead prediction (32). Second, the fixed lag-smoother (32) inherits the stability properties of the original Kalman filter.

## 7.4.2 Reduced-order Solution

Equation (32) is termed a high order solution because the dimension of the above augmented state matrix is  $(N+2)n \times (N+2)n$ . Moore [1] – [3] simplified (32) to obtain elegant reduced order solution structures as follows. Let

$$\begin{bmatrix} P_{k+1/k}^{(0,0)} & P_{k+1/k}^{(0,1)} & \cdots & P_{k+1/k}^{(0,N)} \\ P_{k+1/k}^{(1,0)} & P_{k+1/k}^{(1,1)} & & \vdots \\ \vdots & & \ddots & \\ P_{k+1/k}^{(N,0)} & \cdots & & P_{k+1/k}^{(N,N)} \end{bmatrix}, \quad P_{k+1/k}^{(i,j)} = (P_{k+1/k}^{(j,i)})^T,$$

denote the predicted error covariance matrix. For  $0 \leq i \leq N$ , the smoothed states within (32) are given by

$$\hat{x}_{k-i/k} = \hat{x}_{k-i/k-1} + K_{i+1,k} (z_k - C_k \hat{x}_{k/k-1}), \quad (33)$$

where

$$K_{i+1,k} = P_{k+1/k}^{(i,0)} C_k^T (C_k P_{k+1/k}^{(0,0)} C_k^T + R_k)^{-1}. \quad (34)$$

Recursions for the error covariance submatrices of interest are

$$P_{k+1/k}^{(i+1,0)} = P_{k+1/k}^{(i,0)} (A_k - K_{0,k} C_k)^T, \quad (35)$$

$$P_{k+1/k}^{(i+1,i+1)} = P_{k+1/k}^{(i,i)} - P_{k+1/k}^{(i,0)} C_k K_{i+1,k}^T. \quad (36)$$

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“When I am working on a problem, I never think about beauty but when I have finished, if the solution is not beautiful, I know it is wrong.” *Richard Buckminster Fuller*

Another rearrangement of (33) – (34) to reduce the calculation cost further is described in [1].

### 7.4.3 Performance

Two facts that stem from (36) are stated below.

**Lemma 3:** *In respect of the fixed-lag smoother (33) – (36), the following applies.*

- (i) *The error-performance improves with increasing smoothing lag.*
- (ii) *The fixed-lag smoothers outperform the Kalman filter.*

**Proof:**

- (i) *The claim follows by inspection of (34) and (36).*
- (ii) *The observation follows by recognising that  $P_{k+1/k}^{(1,1)} = E\{(x_k - \hat{x}_{k/k})(x_k - \hat{x}_{k/k})^T\}$  within (i).*

It can also be seen from the term  $-P_{k+1/k}^{(i,0)} C_k^T (C_k P_{k+1/k}^{(0,0)} C_k^T + R_k)^{-1} C_k P_{k+1/k}^{(i,0)}$  within (36) that the benefit of smoothing diminishes as  $R_k$  becomes large.

## 7.5 Fixed-Interval Smoothing

### 7.5.1 The Maximum-Likelihood Smoother

#### 7.5.1.1 Solution Derivation

The most commonly used fixed-interval smoother is undoubtedly the solution reported by Rauch [5] in 1963 and two years later with Tung and Striebel [6]. Although this smoother does not minimise the error variance, it has two desirable attributes. First, it is a low-complexity state estimator. Second, it can provide close to optimal performance whenever the accompanying Gaussian distribution assumptions are reasonable.

The smoother involves two passes. In the first (forward) pass, filtered state estimates,  $\hat{x}_{k/k}$ , are calculated from

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1}), \quad (37)$$

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k}, \quad (38)$$

where  $L_k = P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}$  is the filter gain,  $K_k = A_k L_k$  is the predictor gain, in which  $P_{k/k} = P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1}$  and  $P_{k+1/k} =$

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“Performance is your reality. Forget everything else.” *Harold Sydney Geneen*

$A_k P_{k/k} A_k^T + B_k Q_k B_k^T$ . In the second backward pass, Rauch, Tung and Striebel calculate smoothed state estimates,  $\hat{x}_{k/N}$ , from the elegant one-line recursion

$$\hat{x}_{k/N} = \hat{x}_{k/k} + G_k (\hat{x}_{k+1/N} - \hat{x}_{k+1/k}), \quad (39)$$

where

$$G_k = P_{k/k} A_{k+1}^T P_{k+1/k}^{-1} \quad (40)$$

is the smoother gain. The above sequence is initialised by  $\hat{x}_{k/N} = \hat{x}_{k/k}$  at  $k = N$ . In the first public domain appearance of (39), Rauch [5] referred to a Lockheed Missile and Space Company Technical Report co-authored with Tung and Striebel. Consequently, (39) is commonly known as the Rauch-Tung-Striebel smoother. This smoother was derived in [6] using the maximum-likelihood method and an outline is provided below.

The notation  $x_k \sim \mathcal{N}(\mu, R_{xx})$  means that a discrete-time random variable  $x_k$  with mean  $\mu$  and covariance  $R_{xx}$  has the normal (or Gaussian) probability density function

$$p(x_k) = \frac{1}{(2\pi)^{n/2} |R_{xx}|^{1/2}} \exp\{-0.5(x_k - \mu)^T R_{xx}^{-1}(x_k - \mu)\}. \quad (41)$$

Rauch, Tung and Striebel assumed that [6]:

$$\hat{x}_{k+1/N} \sim \mathcal{N}(A_k \hat{x}_{k/N}, B_k Q_k B_k^T). \quad (42)$$

$$\hat{x}_{k/N} \sim \mathcal{N}(\hat{x}_{k/k}, P_{k/k}). \quad (43)$$

From the approach of [6], setting the partial derivative of the logarithm of the joint density function to zero results in

$$0 = \frac{\partial(\hat{x}_{k+1/N} - A_k \hat{x}_{k/N})^T}{\partial \hat{x}_{k/N}} (B_k Q_k B_k^T)^{-1} (\hat{x}_{k+1/N} - A_k \hat{x}_{k/N})^T \frac{n!}{r!(n-r)!} \\ + \frac{\partial(\hat{x}_{k/N} - \hat{x}_{k/k})^T}{\partial \hat{x}_{k/N}} P_{k/k}^{-1} (\hat{x}_{k/N} - \hat{x}_{k/k}).$$

Rearranging the above equation leads to

$$\hat{x}_{k/N} = (I + P_{k/k} A_k^T (B_k Q_k B_k^T)^{-1} A_k)^{-1} (\hat{x}_{k/N} + P_{k/k} A_k^T B_k Q_k B_k^T)^{-1} \hat{x}_{k/N}). \quad (44)$$

From the Matrix Inversion Lemma

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“An ounce of performance is worth pounds of promises.” *Mary (Mae) Jane West*



$$(I + P_{k/k} A_k^T (B_k Q_k B_k^T)^{-1} A_k)^{-1} = I - G_k A_k. \quad (45)$$

The solution (39) is found by substituting (45) into (44). Some further details of Rauch, Tung and Striebel's derivation appear in [13].

### 7.5.1.2 Alternative Forms

The smoother gain (40) can be calculated in different ways. Assuming that  $A_k$  is non-singular, it follows from  $P_{k+1/k} = A_k P_{k/k} A_k^T + B_k Q_k B_k^T$  that  $P_{k/k} A_{k+1}^T P_{k+1/k}^{-1} = A_k^{-1} (I - B_k Q_k B_k^T P_{k+1/k}^{-1})$  and

$$G_k = A_k^{-1} (I - B_k Q_k B_k^T P_{k+1/k}^{-1}). \quad (46)$$

In applications where difficulties exist with inverting  $P_{k+1/k}$ , it may be preferable to calculate

$$P_{k/k-1}^{-1} = P_{k/k}^{-1} - C_k^T R_k^{-1} C_k. \quad (47)$$

It is shown in [15] that the filter (37) – (38) and the smoother (39) can be written in the following Hamiltonian form

$$\begin{bmatrix} \hat{x}_{k+1/N} \\ \lambda_{k/N} \end{bmatrix} = \begin{bmatrix} A_k & B_k Q_k B_k^T \\ -C_k^T R_k^{-1} C_k & A_k^T \end{bmatrix} \begin{bmatrix} \hat{x}_{k/N} \\ \lambda_{k+1/N} \end{bmatrix} + \begin{bmatrix} 0 \\ C_k^T R_k^{-1} z_k \end{bmatrix}, \quad (48)$$

where  $\lambda_{k/N} \in \mathbb{R}^n$  is an auxiliary variable that proceeds backward in time  $k$ . The form (48) – (49) avoids potential numerical difficulties that may be associated with calculating  $P_{k/k-1}^{-1}$ .

To confirm the equivalence of (39) and (48) – (49), use the Bryson-Frazier formula [15]

$$\hat{x}_{k+1/N} = \hat{x}_{k+1/k+1} + P_{k+1/k} \lambda_{k+1/N}, \quad (50)$$

and (46) within (48) to obtain

$$\hat{x}_{k/N} = G_k \hat{x}_{k+1/k+1} + A_k^{-1} B_k Q_k B_k^T P_{k+1/k}^{-1} \hat{x}_{k+1/k}. \quad (51)$$

Employing (46) within (51) and rearranging leads to (39).

In time-invariant problems, steady state solutions for  $P_{k/k}$  and  $P_{k+1/k}$  can be used to precalculate the gain (40) before running the smoother. For example, the

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“Error is the discipline through which we advance.” *William Ellery Channing*

application of a time-invariant version of the Rauch-Tung-Striebel smoother for the restoration of blurred images is described in [14].

### 7.5.1.3 Performance

An expression for the smoother error covariance is developed below following the approach of [6], [13]. Define the smoother and filter error states as  $\tilde{x}_{k/N} = x_k - \hat{x}_{k/N}$  and  $\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}$ , respectively. It is assumed that

$$E\{\tilde{x}_{k/k} \hat{x}_{k/k}^T\} = 0, \quad (52)$$

$$E\{\tilde{x}_{k+1/N} \hat{x}_{k+1/N}^T\} = 0, \quad (53)$$

$$E\{\tilde{x}_{k+1/N} \hat{x}_{k/N}^T\} = 0. \quad (54)$$

It is straightforward to show that (52) implies

$$E\{\hat{x}_{k+1/k} \hat{x}_{k+1/k}^T\} = E\{x_{k+1} x_{k+1}^T\} - P_{k+1/k}. \quad (55)$$

Denote  $\Sigma_{k/N} = E\{\hat{x}_{k+1/N} \hat{x}_{k+1/N}^T\}$ . The assumption (53) implies

$$E\{\hat{x}_{k+1/N} \hat{x}_{k+1/N}^T\} = E\{x_{k+1} x_{k+1}^T\} - \Sigma_{k+1/N}. \quad (56)$$

Subtracting  $x_k$  from both sides of (39) gives

$$\tilde{x}_{k/N} - G_k \hat{x}_{k+1/N} = \tilde{x}_{k/k} - G_k A_k \hat{x}_{k/k}. \quad (57)$$

By simplifying

$$E\{(\tilde{x}_{k/N} - G_k \hat{x}_{k+1/N})(\tilde{x}_{k/N} - G_k \hat{x}_{k+1/N})^T\} = E\{(\tilde{x}_{k/k} - G_k A_k \hat{x}_{k/k})(\tilde{x}_{k/k} - G_k A_k \hat{x}_{k/k})^T\} \quad (58)$$

and using (52), (54) – (56) yields

$$\Sigma_{k/N} = P_{k/k} - G_k (P_{k+1/k} - \Sigma_{k+1/N}) G_k^T. \quad (59)$$

It can now be shown that the smoother performs better than the Kalman filter.

**Lemma 4:** Suppose that the sequence (59) is initialised with

$$\Sigma_{N+1/N} = P_{N+1/N}, \quad (60)$$

Then  $\Sigma_{k/N} \leq P_{k/k}$  for  $1 \leq k \leq N$ .

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“Great thinkers think inductively, that is, they create solutions and then seek out the problems that the solutions might solve; most companies think deductively, that is, defining a problem and then investigating different solutions.” *Joey Reiman*

**Proof:** The condition (60) implies  $\Sigma_{N/N} = P_{N/N}$ , which is the initial step for an induction argument. For the induction step, (59) is written as

$$\Sigma_{k/N} = P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R)^{-1} C_k P_{k/k-1} - G_k (P_{k+1/k} - \Sigma_{k+1/N}) G_k^T \quad (61)$$

and thus  $\Sigma_{k+1/N} \leq P_{k+1/k}$  implies  $\Sigma_{k/N} \leq P_{k/k-1}$  and  $\Sigma_{k/N} \leq P_{k/k}$ .

### 7.5.2 The Fraser-Potter Smoother

Forward and backward estimates may be merged using the data fusion formula described in Lemma 6. A variation of the Fraser-Potter discrete-time fixed-interval smoother [4] derived by Monzingo [16] is advocated below.

In the first pass, a Kalman filter produces corrected state estimates  $\hat{x}_{k/k}$  and corrected error covariances  $P_{k/k}$  from the measurements. In the second pass, a Kalman filter is employed to calculate predicted “backward” state estimates  $\hat{x}_{k-1/k}$  and predicted “backward” error covariances  $\Sigma_{k-1/k}$  from the time-reversed measurements. The smoothed estimate is given by [16]

$$\hat{x}_{k/N} = (P_{k/k}^{-1} + \Sigma_{k-1/k}^{-1})^{-1} (P_{k/k}^{-1} \hat{x}_{k/k} + \Sigma_{k-1/k}^{-1} \hat{x}_{k-1/k}). \quad (62)$$

Alternatively, Kalman filters could be used to derive predicted quantities,  $\hat{x}_{k/k-1}$  and  $P_{k/k-1}$ , from the measurements, and backward corrected quantities  $\hat{x}_{k/k}$  and  $\Sigma_{k/k}$ . Smoothed estimates may then be obtained from the linear combination

$$\hat{x}_{k/N} = (P_{k/k-1}^{-1} + \Sigma_{k/k}^{-1})^{-1} (P_{k/k-1}^{-1} \hat{x}_{k/k-1} + \Sigma_{k/k}^{-1} \hat{x}_{k/k}). \quad (63)$$

It is observed that the fixed-point smoother (24), the fixed-lag smoother (32), maximum-likelihood smoother (39), the smoothed estimates (62) – (63) and the minimum-variance smoother (which is described subsequently) all use each measurement  $z_k$  once.

Note that Fraser and Potter’s original smoother solution [4] and Monzingo’s variation [16] are *ad hoc* and no claims are made about attaining a prescribed level of performance.

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“No great discovery was ever made without a bold guess.” *Isaac Newton*

## 7.5.3 Minimum-Variance Smoothers

### 7.5.3.1 Optimal Unrealisable Solutions

Consider again the estimation problem depicted in Fig. 1 of Chapter 6, where  $w$  and  $v$  are now discrete-time inputs. As in continuous-time, it is desired to construct a solution  $\mathcal{H}$  is that produces output estimates  $\hat{y}_1$  of a reference system  $y_1 = \mathcal{G}_1 w$  from observations  $z = y_2 + v$ , where  $y_2 = \mathcal{G}_2 w$ . The objective is to minimise the energy of the output estimation error  $\tilde{y} = y_1 - \hat{y}_1$ .

The following discussion is perfunctory since it is a regathering of the results from Chapter 6. Recall that the output estimation error is generated by  $\tilde{y} = \mathcal{R}_{yi} i$ , where

$$\mathcal{R}_{yi} = -[\mathcal{H} \quad \mathcal{H}\mathcal{G}_2 - \mathcal{G}_1] \text{ and } i = \begin{bmatrix} v \\ w \end{bmatrix}. \text{ It has been shown previously that } \mathcal{R}_{yi} \mathcal{R}_{yi}^H = \mathcal{R}_{yi1} \mathcal{R}_{yi1}^H + \mathcal{R}_{yi2} \mathcal{R}_{yi2}^H \text{ where}$$

$$\mathcal{R}_{yi2} = \mathcal{H}\Delta - \mathcal{G}_1 \mathcal{Q} \mathcal{G}_2^H \Delta^{-H}, \quad (64)$$

in which  $\Delta$  is known as the Wiener-Hopf factor, which satisfies  $\Delta \Delta^H = \mathcal{G} \mathcal{Q} \mathcal{G}^H + R$ . From Lemma 8 of Chapter 6, the smoother solution  $\mathcal{H} = \mathcal{G}_2 \mathcal{Q} \mathcal{G}_2^H (\Delta \Delta^H)^{-1}$  achieves the best-possible performance, namely, it minimises  $\|\tilde{y} \tilde{y}^H\|_2 = \|\mathcal{R}_{yi} \mathcal{R}_{yi}^H\|_2$ . For example, in output estimation problems  $\mathcal{G}_1 = \mathcal{G}_2$  and the optimal smoother simplifies to  $\mathcal{H}_{OE} = I - R(\Delta \Delta^H)^{-1}$ . From Lemma 9 of Chapter 6, the (causal) filter solution  $\mathcal{H}_{OE} = \{\mathcal{G}_2 \mathcal{Q} \mathcal{G}_2^H \Delta^{-H} \Delta^{-1}\}_+ = \{\mathcal{G}_2 \mathcal{Q} \mathcal{G}_2^H \Delta^{-H}\}_+ \Delta^{-1}$  achieves the best-possible filter performance, that is, it minimises  $\|\{\tilde{y}_2 \tilde{y}_2^H\}_+\|_2 = \|\{\mathcal{R}_{yi} \mathcal{R}_{yi}^H\}_+\|_2$ . The optimal smoother outperforms the optimal filter since  $\|\{\mathcal{R}_{yi} \mathcal{R}_{yi}^H\}_+\|_2 \geq \|\mathcal{R}_{yi} \mathcal{R}_{yi}^H\|_2$ . The above solutions are termed unrealisable because of the difficulty in obtaining  $\Delta$  when  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are time-varying systems. Realisable solutions that use an approximate Wiener-Hopf factor in place of  $\Delta$  are presented below.

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“Life is pretty simple. You do stuff. Most fails. Some works. You do more of what works. If it works big, others quickly copy it. Then you do something else. The trick is the doing something else.”  
Leonardo da Vinci

### 7.5.3.2 Non-causal Output Estimation

Suppose that the time-varying linear system  $\mathcal{G}_2$  has the realisation (1) – (2). An approximate Wiener-Hopf factor  $\hat{\Delta}$  is introduced in [7], [13] and defined by

$$\begin{bmatrix} x_{k+1} \\ \delta_k \end{bmatrix} = \begin{bmatrix} A_k & K_k \Omega_k^{1/2} \\ C_k & \Omega_k^{1/2} \end{bmatrix} \begin{bmatrix} x_k \\ z_k \end{bmatrix}, \quad (65)$$

where  $K_k = (A_k P_{k/k-1} C_k^T + B_k Q_k D_k^T) \Omega_k^{-1}$  is the predictor gain,  $\Omega_k = C_k P_{k/k-1} C_k^T + D_k Q_k D_k^T + R_k$  and  $P_{k/k-1}$  evolves from the Riccati difference equation  $P_{k+1/k} = A_k P_{k/k-1} A_k^T - (A_k P_{k/k-1} C_k^T + B_k Q_k D_k^T)(C_k P_{k/k-1} C_k^T + D_k Q_k D_k^T + R_k)^{-1}(C_k P_{k/k-1} A_k^T + D_k Q_k B_k^T) + B_k Q_k B_k^T$ . The inverse of the system (65), denoted by  $\hat{\Delta}^{-1}$ , is obtained using the Matrix Inversion Lemma

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \alpha_k \end{bmatrix} = \begin{bmatrix} A_k - K_k C_k & K_k \\ -\Omega_k^{-1/2} C_k & \Omega_k^{-1/2} \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix}. \quad (66)$$

The optimal output estimation smoother can be approximated as

$$\begin{aligned} \mathcal{H}_{OE} &= I - R(\hat{\Delta} \hat{\Delta}^H)^{-1} \\ &= I - R \hat{\Delta}^{-H} \hat{\Delta}^{-1}. \end{aligned} \quad (67)$$

A state-space realisation of (67) is given by (66),

$$\begin{bmatrix} \xi_{k-1} \\ \beta_k \end{bmatrix} = \begin{bmatrix} A_k^T - C_k^T K_k^T & C_k^T \Omega_k^{-1/2} \\ -K_k^T & \Omega_k^{-1/2} \end{bmatrix} \begin{bmatrix} \xi_k \\ \alpha_k \end{bmatrix}, \quad \xi_N = 0 \quad (68)$$

and

$$\hat{y}_{k/N} = z_k - R_k \beta_k. \quad (69)$$

Note that Lemma 1 is used to obtain the realisation (68) of  $\hat{\Delta}^{-H} = (\hat{\Delta}^H)^{-1}$  from (66). A block diagram of this smoother is provided in Fig. 2. The states  $\hat{x}_{k/k-1}$  within (66) are immediately recognisable as belonging to the one-step-ahead predictor. Thus, the optimum realisable solution involves a cascade of familiar building blocks, namely, a Kalman predictor and its adjoint.

**Procedure 1.** The above output estimation smoother can be implemented via the following three-step procedure.

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“When I examine myself and my methods of thought, I come to the conclusion that the gift of fantasy has meant more to me than any talent for abstract, positive thinking.” *Albert Einstein*

- Step 1. Operate  $\hat{\Delta}^{-1}$  on  $z_k$  using (66) to obtain  $\alpha_k$ .
- Step 2. In lieu of the adjoint system (68), operate (66) on the time-reversed transpose of  $\alpha_k$ . Then take the time-reversed transpose of the result to obtain  $\beta_k$ .
- Step 3. Calculate the smoothed output estimate from (69).  
It is shown below that  $\hat{y}_{k/N}$  is an unbiased estimate of  $y_k$ .

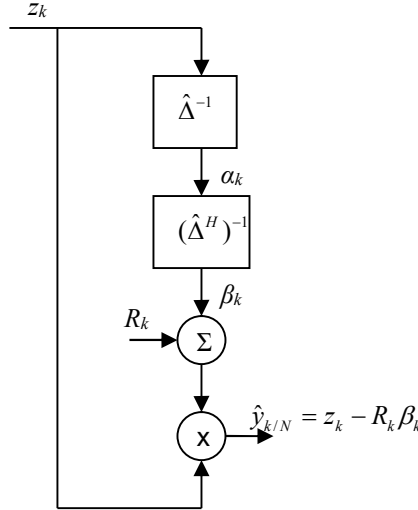


Fig. 2. Block diagram of the output estimation smoother

**Lemma 5**  $E\{\hat{y}_{k/N}\} = E\{y_k\}$ .

**Proof:** Denote the one-step-head prediction error by  $\tilde{x}_{k/k-1} = x_k - \hat{x}_{k/k-1}$ . The output of (66) may be written as  $\alpha_k = \Omega_k^{-1/2} C_k \tilde{x}_{k/k-1} + \Omega_k^{-1/2} v_k$  and therefore

$$E\{\alpha_k\} = \Omega_k^{-1/2} C_k E\{\tilde{x}_{k/k-1}\} + \Omega_k^{-1/2} E\{v_k\}. \quad (70)$$

The first term on the right-hand-side of (70) is zero since it pertains to the prediction error of the Kalman filter. The second term is zero since it is assumed that  $E\{v_k\} = 0$ . Thus  $E\{\alpha_k\} = 0$ . Since the recursion (68) is initialised with  $\zeta_N = 0$ , it follows that  $E\{\zeta_k\} = 0$ , which implies  $E\{\zeta_k\} = -K_k E\{\zeta_k\} + \Omega_k^{-1/2} E\{\alpha_k\} = 0$ . Thus, from (69),  $E\{\hat{y}_{k/N}\} = E\{z_k\} = E\{y_k\}$ , since it is assumed that  $E\{v_k\} = 0$ .

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“The practical success of an idea, irrespective of its inherent merit, is dependent on the attitude of the contemporaries. If timely it is quickly adopted; if not, it is apt to fare like a sprout lured out of the ground by warm sunshine, only to be injured and retarded in its growth by the succeeding frost.”  
Nicola Tesla

### 7.5.3.3 Causal Output Estimation

The minimum-variance (Kalman) filter is obtained by taking the causal part of the optimum minimum-variance smoother (67)

$$\begin{aligned}\{\mathcal{H}_{OE}\}_+ &= I - R_k \{\hat{\Delta}^{-H}\}_+ \hat{\Delta}^{-1} \\ &= I - R_k \Omega_k^{-1/2} \hat{\Delta}^{-1}.\end{aligned}\quad (71)$$

To confirm this linkage between the smoother and filter, denote  $L_k = (C_k P_{k/k-1} C_k^T + D_k Q_k D_k^T) \Omega_k^{-1}$  and use (71) to obtain

$$\begin{aligned}\hat{y}_{k/k} &= z_k - R_k \Omega_k^{-1/2} \alpha_k \\ &= R_k \Omega_k^{-1} C_k x_k + (I - R_k \Omega_k^{-1/2}) z_k \\ &= (C_k - L_k C_k) x_k + L_k z_k,\end{aligned}\quad (72)$$

which is identical to (34) of Chapter 4.

### 7.5.3.4 Input Estimation

As discussed in Chapter 6, the optimal realisable smoother for input estimation is

$$\mathcal{H}_{IE} = Q \mathcal{G}_2^H \hat{\Delta}^{-H} \hat{\Delta}^{-1}.\quad (73)$$

The development of a state-space realisation for  $\hat{w}_{k/N} = Q \mathcal{G}_2^H \hat{\Delta}^{-H} \alpha_k$  makes use of the formula for the cascade of two systems described in Chapter 6. The smoothed input estimate is realised by

$$\begin{bmatrix} \xi_{k-1} \\ \gamma_{k-1} \\ \hat{w}_{k-1/N} \end{bmatrix} = \begin{bmatrix} A_k^T - C_k^T K_k & 0 & C_k^T \Omega_k^{-1/2} \\ C_k^T K_k^T & A_k^T & -C_k^T \Omega_k^{-1/2} \\ -Q_k D_k^T K_k^T & -Q_k B_k^T & Q_k D_k^T \Omega_k^{-1/2} \end{bmatrix} \begin{bmatrix} \xi_k \\ \gamma_k \\ \alpha_k \end{bmatrix},\quad (74)$$

in which  $\gamma_k \in \mathbb{R}^n$  is an auxiliary state.

**Procedure 2.** The above input estimator can be implemented via the following three steps.

- Step 1. Operate  $\hat{\Delta}^{-1}$  on the measurements  $z_k$  using (66) to obtain  $\alpha_k$ .
- Step 2. Operate the adjoint of (74) on the time-reversed transpose of  $\alpha_k$ . Then take the time-reversed transpose of the result.

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“Doubt is the father of invention.” *Galileo Galilei*

### 7.5.3.5 State Estimation

Smoothed state estimates can be obtained by defining the reference system  $\mathcal{G}_1 = I$  which yields

$$\begin{aligned}\hat{x}_{k+1/N} &= A_k \hat{x}_{k/N} + B_k \hat{w}_{k/N} \\ &= A_k \hat{x}_{k/N} + B_k Q \mathcal{G}_2^H \hat{\Delta}^{-H} \alpha_k.\end{aligned}\quad (75)$$

Thus, the minimum-variance smoother for state estimation is given by (66) and (74) – (75). As remarked in Chapter 6, some numerical model order reduction may be required. In the special case of  $C_k$  being of rank  $n$  and  $D_k = 0$ , state estimates can be calculated from (69) and

$$\hat{x}_{k/N} = C_k^\dagger \hat{y}_{k/N} . \quad (76)$$

where  $C_k^\dagger = (C_k^T C_k)^{-1} C_k^T$  is the Moore-Penrose pseudo-inverse.

### 7.5.3.6 Performance

The characterisation of smoother performance requires the following additional notation. Let  $\gamma = \mathcal{G}_0 w$  denote the output of the linear time-varying system having the realisation

$$x_{k+1} = A_k x_k + w_k , \quad (77)$$

$$\gamma_k = x_k , \quad (78)$$

where  $A_k \in \mathbb{R}^{n \times n}$ . By inspection of (77) – (78), the output of the inverse system  $w = \mathcal{G}_0^{-1} \gamma$  is given by

$$w_k = \gamma_{k+1} - A_k \gamma_k . \quad (79)$$

Similarly, let  $\varepsilon = \mathcal{G}_0^H u$  denote the output of the adjoint system  $\mathcal{G}_0^H$ , which from Lemma 1 has the realisation

$$\zeta_{k-1} = A_k^T \zeta_k - u_k , \quad (80)$$

$$\varepsilon_k = -\zeta_k . \quad (81)$$

It follows that the output of the inverse system  $u = \mathcal{G}_0^{-H} \varepsilon$  is given by

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“The theory of our modern technique shows that nothing is as practical as the theory.” *Julius Robert Oppenheimer*



$$u_k = \varepsilon_{k-1} - A_k^T \varepsilon_k. \quad (82)$$

The exact Wiener-Hopf factor may now be written as

$$\Delta \Delta^H = C_k \mathcal{G}_0 B_k Q_k B_k^T \mathcal{G}_0^H C_k^T + R_k. \quad (83)$$

The subsequent lemma, which relates the exact and approximate Wiener-Hopf factors, requires the identity

$$P_k - A_k P_k A_k^T = A_k P_k \mathcal{G}_0^{-H} + \mathcal{G}_0^{-1} P_k A_k^T + \mathcal{G}_0^{-1} P_k \mathcal{G}_0^{-H}, \quad (84)$$

in which  $P_k$  is an arbitrary matrix of appropriate dimensions. A verification of (84) is requested in the problems.

**Lemma 6 [7]:** In respect of the signal model (1) – (2) with  $D_k = 0$ ,  $E\{w_k\} = E\{v_k\} = 0$ ,  $E\{w_j w_k^T\} = Q_k \delta_{jk}$ ,  $E\{v_j v_k^T\} = R_k \delta_{jk}$ ,  $E\{w_k v_k^T\} = 0$  and the quantities defined above,

$$\hat{\Delta} \hat{\Delta}^H = \Delta \Delta^H + C_k \mathcal{G}_0 (P_{k/k-1} - P_{k+1/k}) \mathcal{G}_0^H C_k^T. \quad (85)$$

**Proof:** The approximate Wiener-Hopf factor may be written as  $\hat{\Delta} = C_k \mathcal{G}_0 K_k \Omega_k^{1/2} + \Omega_k^{1/2}$ . Using  $P_{k/k-1} - A_k P_{k/k-1} A_k^T = -A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1} A_k^T + B_k Q_k B_k^T + P_{k/k-1} - P_{k+1/k}$  within (84) and simplifying leads to (85).

It can be seen from (85) that the approximate Wiener-Hopf factor approaches the exact Wiener-Hopf factor whenever the estimation problem is locally stationary, that is, when the model and noise parameters vary sufficiently slowly so that  $P_{k/k-1} \approx P_{k+1/k}$ . Under these conditions, the smoother (69) achieves the best-possible performance, as is shown below.

**Lemma 7 [7]:** The smoother (67) satisfies

$$\mathcal{R}_{ei2} = R_k [(\Delta \Delta^H)^{-1} - (\Delta \Delta^H - C_k \mathcal{G}_0 (P_{k/k-1} - P_{k+1/k}) \mathcal{G}_0^H C_k^T)^{-1}] \Delta, \quad (86)$$

**Proof:** Substituting (67) into (64) yields

$$\mathcal{R}_{ei2} = R_k [(\Delta \Delta^H)^{-1} - (\hat{\Delta} \hat{\Delta}^H)^{-1}] \Delta. \quad (87)$$

The result (86) is now immediate from (85) and (87).

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“Whoever, in the pursuit of science, seeks after immediate practical utility may rest assured that he seeks in vain.” *Hermann Ludwig Ferdinand von Helmholtz*

Some conditions under which  $P_{k+1/k}$  asymptotically approaches  $P_{k/k-1}$  and the smoother (67) attaining optimal performance are set out below.

**Lemma 8 [8]:** Suppose

- (i) for a  $t > 0$  that there exist solutions  $P_t \geq P_{t+1} \geq 0$  of the Riccati difference equation
- (ii)  $P_{k+1/k} = A_k P_{k/k-1} A_k^T - A_k P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1} A_k^T + B_k Q_k B_k^T$ ; and
- (iii) 
$$\begin{bmatrix} B_{t+k-1} Q_{t+k-1} B_{t+k-1}^T & A_{t+k-1}^T \\ A_{t+k-1} & -C_{t+k-1}^T R_{t+k-1}^{-1} C_{t+k-1} \end{bmatrix} \geq \begin{bmatrix} B_{t+k} Q_{t+k} B_{t+k}^T & A_{t+k}^T \\ A_{t+k} & -C_{t+k}^T R_{t+k}^{-1} C_{t+k} \end{bmatrix}$$
 for all  $k \geq 0$ .

Then the smoother (67) achieves

$$\lim_{t \rightarrow \infty} \left\| \mathcal{R}_{ei2} \mathcal{R}_{ei2}^H \right\|_2 = 0. \quad (88)$$

**Proof:** Conditions i) and ii) together with Theorem 1 imply  $P_{k/k-1} \geq P_{k+1/k}$  for all  $k \geq 0$  and  $P_{k/k-1} = 0$ . The claim (88) is now immediate from Theorem 2.

An example that illustrates the performance benefit of the minimum-variance smoother (69) is described below.

**Example 2 [9].** The nominal drift rate of high quality inertial navigation systems is around one nautical mile per hour, which corresponds to position errors of about 617 m over a twenty minute period. Thus, inertial navigation systems alone cannot be used to control underground mining equipment. An approach which has been found to be successful in underground mines is called dead reckoning, where the Euler angles,  $\theta_k$ ,  $\phi_k$  and  $\psi_k$ , reported by an inertial navigation system are combined with external odometer measurements,  $d_k$ . The dead reckoning position estimates in the x-y-z plane are calculated as

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} + (d_k - d_{k-1}) \begin{bmatrix} \sin(\theta_k) \\ \sin(\phi_k) \\ \sin(\psi_k) \end{bmatrix}. \quad (89)$$

A filter or a smoother may then be employed to improve the noisy position estimates calculated from (89). Euler angles were generated using

$$\begin{bmatrix} \theta_{k+1} \\ \phi_{k+1} \\ \psi_{k+1} \end{bmatrix} = A \begin{bmatrix} \theta_k \\ \phi_k \\ \psi_k \end{bmatrix} + \begin{bmatrix} w_k^{(1)} \\ w_k^{(2)} \\ w_k^{(3)} \end{bmatrix}, \text{ with } A = \begin{bmatrix} 0.95 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.95 \end{bmatrix} \text{ and } w_k^{(i)} \sim \mathcal{N}(0, 0.01), i$$

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“I do not think that the wireless waves that I have discovered will have any practical application.”  
Heinrich Rudolf Hertz

= 1...3. Simulations were conducted with 1000 realisations of Gaussian measurement noise added to position estimates calculated from (89). The mean-square error exhibited by the minimum-variance filter and smoother operating on the noisy dead reckoning estimates are shown in Fig. 3. It can be seen that filtering the noisy dead reckoning positions can provide a significant mean-square-error improvement. The figure also demonstrates that the smoother can offer a few dB of further improvement at mid-range signal-to-noise ratios.

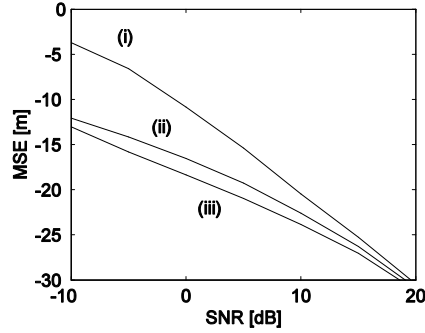


Fig. 3. Mean-square-error of the position estimate versus input signal to noise ratio for Example 2: (i) noisy dead reckoning data, (ii) Kalman filter, and (iii) minimum-variance smoother (69)

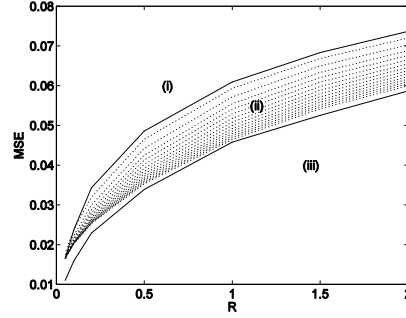


Fig. 4. Mean-square-error versus measurement noise covariance for Example 3: (i) Kalman filter, (ii) fixed-lag smoothers, and (iii) optimal minimum-variance smoother (69).

#### 7.5.4 Performance Comparison

It has been demonstrated by the previous examples that the optimal fixed-interval smoother provides a performance improvement over the maximum-likelihood smoother. The remaining example of this section compares the behaviour of the fixed-lag and the optimum fixed-interval smoother.

**Example 3.** Simulations were conducted for a first-order output estimation problem, in which  $A = 0.95$ ,  $B = 1$ ,  $C = 0.1$ ,  $Q = 1$ ,  $R = \{0.01, 0.02, 0.5, 1, 1.5, 2\}$  and  $N = 20,000$ . The mean-square-errors exhibited by the Kalman filter and the optimum fixed-interval smoother (69) are indicated by the top and bottom solid lines of Fig. 4, respectively. Fourteen fixed-lag smoother output error covariances,  $CP_{k+1/k}^{(i+1,i+1)}C^T$ ,  $i = 2 \dots 15$ , were calculated from (35) – (36) and are indicated by the dotted lines of Fig. 4. The figure illustrates that the fixed-lag smoother mean-square errors are bounded above and below by those of the Kalman filter and optimal fixed-interval smoother, respectively. Thus, an option for asymptotically

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“You see, wire telegraph is a kind of a very, very long cat. You pull his tail in New York and his head is meowing in Los Angeles. Do you understand this? And radio operates exactly the same way: you send signals here, they receive them there. The only difference is that there is no cat.” *Albert Einstein*

attaining optimal performance is to employ Moore's reduced-order fixed-lag solution [1] – [3] with a sufficiently long lag.

## 7.6 Chapter Summary

Solutions to the fixed-point and fixed-lag smoothing problems can be found by applying the standard Kalman filter recursions to augmented systems. Where possible, it is shown that the smoother error covariances are less than the filter error covariance, namely, the fixed-point and fixed-lag smoothers provide a performance improvement over the filter.

	ASSUMPTIONS	MAIN RESULTS
Signals and system	$E\{w_k\} = E\{v_k\} = 0$ . $E\{w_k w_k^T\} = Q_k > 0$ and $E\{v_k v_k^T\} = R_k > 0$ are known. $A_k$ , $B_k$ , $C_{1,k}$ and $C_{2,k}$ are known.	$x_{k+1} = A_k x_k + B_k w_k$ $y_{2,k} = C_{2,k} x_k$ $z_k = y_{2,k} + v_k$ $y_{1,k} = C_{1,k} x_k$
RTS smoother	Assumes that the filtered and smoothed states are normally distributed. $\hat{x}_{k+1/k}$ previously calculated by Kalman filter.	$\hat{x}_{k/N} = \hat{x}_{k/k} + G_k (\hat{x}_{k+1/N} - \hat{x}_{k+1/k})$ $\hat{y}_{1,k/N} = C_{1,k} \hat{x}_{k/N}$
FP smoother	$\hat{x}_{k/k}$ and $\xi_{k/k-1}$ previously calculated by forward and backward Kalman filters.	$\hat{x}_{k/N} = (P_{k/k}^{-1} + \Sigma_{k/k-1}^{-1})^{-1} (P_{k/k}^{-1} \hat{x}_{k/k} + \Sigma_{k/k-1}^{-1} \xi_{k/k-1})$ $\hat{y}_{1,k/N} = C_{1,k} \hat{x}_{k/N}$
Optimal minimum-		$\begin{bmatrix} x_{k+1/k} \\ \alpha_k \end{bmatrix} = \begin{bmatrix} A_k - K_k C_{2,k} & K_k \\ -\Omega_k^{-1/2} C_{2,k} & \Omega_k^{-1/2} \end{bmatrix} \begin{bmatrix} x_{k/k-1} \\ z_k \end{bmatrix}$ $\begin{bmatrix} \xi_{k-1} \\ \beta_k \end{bmatrix} = \begin{bmatrix} A_k^T - C_{2,k}^T K_k^T & C_{2,k}^T \Omega_k^{-1/2} \\ -K_k^T & \Omega_k^{-1/2} \end{bmatrix} \begin{bmatrix} \xi_k \\ \alpha_k \end{bmatrix}$ $\hat{y}_{2,k/N} = z_k - R_k \beta_k$

Table 1. Main results for discrete-time fixed-interval smoothing.

Table 1 summarises three common fixed-interval smoothers that operate on measurements  $z_k = y_{2,k} + v_k$  of a system  $\mathcal{G}_2$  realised by  $x_{k+1} = A_k x_k + B_k w_k$  and  $y_{2,k} =$

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“My invention, (the motion picture camera), can be exploited for a certain time as a scientific curiosity, but apart from that it has no commercial value whatsoever.” *August Marie Louis Nicolas Lumiere*

$C_{2,k}x_k$ . Monzingo modified the Fraser-Potter smoother solution so that each measurement is only used once. Rauch, Tung and Striebel employed the maximum-likelihood method to derive their fixed-interval smoother in which  $G_k = P_{k/k} A_{k+1}^T P_{k+1/k}^{-1}$  is a gain matrix. Although this is not a minimum-mean-square-error solution, it outperforms the Kalman filter and can provide close to optimal performance whenever the underlying Gaussian-distribution assumptions are reasonable.

The minimum-variance smoothers are state-space generalisations of the optimal noncausal Wiener solutions. They make use of the inverse of the approximate Wiener-Hopf factor  $\hat{\Delta}^{-1}$  and its adjoint  $\hat{\Delta}^{-H}$ . These smoothers achieve the best-possible performance, however, they are not minimum-order solutions. Consequently, any performance benefits need to be reconciled against their increased complexity.

## 7.7 Problems

### Problem 1.

- (i) Simplify the fixed-lag smoother

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \hat{x}_{k/k} \\ \hat{x}_{k-1/k} \\ \vdots \\ \hat{x}_{k-N+1/k} \end{bmatrix} = \begin{bmatrix} A_k & 0 & \cdots & 0 \\ I_n & 0 & & 0 \\ 0 & I_n & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & I_n \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ \hat{x}_{k-1/k-1} \\ \hat{x}_{k-2/k-1} \\ \vdots \\ \hat{x}_{k-N/k-1} \end{bmatrix} + \begin{bmatrix} K_{0,k} \\ K_{1,k} \\ K_{2,k} \\ \vdots \\ K_{N,k} \end{bmatrix} (z_k - C_k \hat{x}_{k/k-1})$$

to obtain an expression for the components of the smoothed state.

- (ii) Derive expressions for the two predicted error covariance submatrices of interest.

### Problem 2.

- (i) With the quantities defined in Section 4 and the assumptions  $\hat{x}_{k+1/N} \sim \mathcal{N}(A_k \hat{x}_{k/N}, B_k Q_k B_k^T)$ ,  $\hat{x}_{k/N} \sim \mathcal{N}(\hat{x}_{k/k}, P_{k/k})$ , use the maximum-likelihood method to derive

$$\hat{x}_{k/N} = (I + P_{k/k} A_k^T (B_k Q_k B_k^T)^{-1} A_k)^{-1} (\hat{x}_{k/N} + P_{k/k} A_k^T B_k Q_k B_k^T)^{-1} \hat{x}_{k/N}^T.$$

- (i) Use the Matrix Inversion Lemma to obtain Rauch, Tung and Striebel's smoother

$$\hat{x}_{k/N} = \hat{x}_{k/k} + G_k (\hat{x}_{k+1/N} - \hat{x}_{k+1/k}).$$

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“What makes photography a strange invention is that its primary raw materials are light and time.”  
John Berger

(ii) Employ the additional assumptions  $E\{\tilde{x}_{k/k}\hat{x}_{k/k}^T\} = 0$ ,  $E\{\tilde{x}_{k+1/N}\hat{x}_{k+1/N}^T\}$  and  $E\{\tilde{x}_{k+1/N}\hat{x}_{k/N}^T\}$  to show that  $E\{\hat{x}_{k+1/k}\hat{x}_{k+1/k}^T\} = E\{x_{k+1}x_{k+1}^T\} - P_{k+1/k}$ ,  $E\{\hat{x}_{k+1/N}\hat{x}_{k+1/N}^T\} = E\{x_{k+1}x_{k+1}^T\} - \Sigma_{k+1/N}$  and  $\Sigma_{k/N} = P_{k/k} - G_k(P_{k+1/k} - \Sigma_{k+1/N})G_k^T$ .

(iii) Use  $G_k = A_k^{-1}(I - B_k Q_k B_k^T P_{k+1/k}^{-1})$  and  $\hat{x}_{k+1/N} = \hat{x}_{k+1/k+1} + P_{k+1/k} \lambda_{k+1/N}$  to confirm that the smoothed estimate within

$$\begin{bmatrix} \hat{x}_{k+1/N} \\ \lambda_{k/N} \end{bmatrix} = \begin{bmatrix} A_k & B_k Q_k B_k^T \\ -C_k^T R_k^{-1} C_k & A_k^T \end{bmatrix} \begin{bmatrix} \hat{x}_{k+1/N} \\ \lambda_{k+1/N} \end{bmatrix} + \begin{bmatrix} 0 \\ C_k^T R_k^{-1} z_k \end{bmatrix}$$

is equivalent to Rauch, Tung and Striebel's maximum-likelihood solution.

**Problem 3.** Let  $\alpha = \mathcal{G}_0 w$  denote the output of linear time-varying system having the realisation  $x_{k+1} = A_k x_k + w_k$ ,  $y_k = x_k$ . Verify that  $P_k - A_k P_k A_k^T = A_k P_k \mathcal{G}_0^{-H} + \mathcal{G}_0^{-1} P_k A_k^T + \mathcal{G}_0^{-1} P_k \mathcal{G}_0^{-H}$ .

**Problem 4.** For the model (1) – (2). assume that  $D_k = 0$ ,  $E\{w_k\} = E\{v_k\} = 0$ ,  $E\{w_k w_k^T\} = Q_k \delta_{jk}$ ,  $E\{v_j v_k^T\} = R_k \delta_{jk}$  and  $E\{w_k v_k^T\} = 0$ . Use the result of Problem 3 to show that  $\hat{\Delta} \hat{\Delta}^H = \Delta \Delta^H - C_k \mathcal{G}_0 (P_{k/k-1} - P_{k+1/k}) \mathcal{G}_0^H C_k^T$ .

**Problem 5.** Under the assumptions of Problem 4, obtain an expression relating  $\hat{\Delta} \hat{\Delta}^H$  and  $\Delta \Delta^H$  for the case where  $D_k \neq 0$ .

**Problem 6.** From  $\mathcal{R}_{ei} = -[\mathcal{H} \quad \mathcal{H} \mathcal{G}_2 - \mathcal{G}_1]$ ,  $\mathcal{R}_{ei} \mathcal{R}_{ei}^H = \mathcal{R}_{ei1} \mathcal{R}_{ei1}^H + \mathcal{R}_{ei2} \mathcal{R}_{ei2}^H$  and  $\mathcal{R}_{ei2} = \mathcal{H} \Delta - \mathcal{G}_1 Q \mathcal{G}_2^H \Delta^{-H}$ , obtain the optimal realisable smoother solutions for output estimation, input estimation and state estimation problems.

## 7.8 Glossary

$p(x_k)$	Probability density function of a discrete random variable $x_k$ .
$x_k \sim \mathcal{N}(\mu, R_{xx})$	The random variable $x_k$ has a normal distribution with mean $\mu$ and covariance $R_{xx}$ .
$\Omega_{k/k}$	Error covariance of the fixed-point smoother.
$P_{k+1/k}^{(i,i)}$	Error covariance of the fixed-lag smoother.
$\hat{w}_{k/N}, \hat{x}_{k/N}, \hat{y}_{k/N}$	Estimates of $w_k$ , $x_k$ and $y_k$ at time $k$ , given data $z_k$ over an

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“To invent, you need a good imagination and a pile of junk.” *Thomas Alva Edison*

	interval $k \in [0, N]$ .
$\bar{G}_k$	Gain of the smoother developed by Rauch, Tung and Striebel.
$\alpha_k$	Output of $\hat{\Delta}^{-1}$ , the inverse of the approximate Wiener-Hopf factor.
$\beta_k$	Output of $\hat{\Delta}^{-H}$ , the adjoint of the inverse of the approximate Wiener-Hopf factor.
$C_k^\#$	Moore-Penrose pseudoinverse of $C_k$ .
$\mathcal{R}_{ei}$	A system (or map) that operates on the problem inputs $i = \begin{bmatrix} v \\ w \end{bmatrix}$ to produce an estimation error $e$ . It is convenient to make use of the factorisation $\mathcal{R}_{ei}\mathcal{R}_{ei}^H = \mathcal{R}_{ei1}\mathcal{R}_{ei1}^H + \mathcal{R}_{ei2}\mathcal{R}_{ei2}^H$ , where $\mathcal{R}_{ei2}\mathcal{R}_{ei2}^H$ includes the filter/smoothing solution and $\mathcal{R}_{ei1}\mathcal{R}_{ei1}^H$ is a lower performance bound.

## 7.9 References

- [1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1979.
- [2] J. B. Moore, "Fixed-Lag Smoothing Results for Linear Dynamical Systems", *A.T.R.*, vol. 7, no. 2, pp. 16 – 21, 1973.
- [3] J. B. Moore, "Discrete-Time Fixed-Lag Smoothing Algorithms", *Automatica*, vol. 9, pp. 163 – 173, 1973.
- [4] D. C. Fraser and J. E. Potter, "The Optimum Linear Smoother as a Combination of Two Optimum Linear Filters", *IEEE Transactions on Automatic Control*, vol. AC-14, no. 4, pp. 387 – 390, Aug., 1969.
- [5] H. E. Rauch, "Solutions to the Linear Smoothing Problem", *IEEE Transactions on Automatic Control*, vol. 8, no. 4, pp. 371 – 372, Oct. 1963.
- [6] H. E. Rauch, F. Tung and C. T. Striebel, "Maximum Likelihood Estimates of Linear Dynamic Systems", *AIAA Journal*, vol. 3, no. 8, pp. 1445 – 1450, Aug., 1965.
- [7] G. A. Einicke, "Optimal and Robust Noncausal Filter Formulations", *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 1069 - 1077, Mar. 2006.
- [8] G. A. Einicke, "Asymptotic Optimality of the Minimum Variance Fixed-Interval Smoother", *IEEE Transactions on Signal Processing*, vol. 55, no. 4, pp. 1543 – 1547, Apr. 2007.
- [9] G. A. Einicke, J. C. Ralston, C. O. Hargrave, D. C. Reid and D. W. Hainsworth, "Longwall Mining Automation, An Application of Minimum-Variance Smoothing", *IEEE Control Systems Magazine*, vol.

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"The cinema is little more than a fad. It's canned drama. What audiences really want to see is flesh and blood on the stage." *Charles Spencer (Charlie) Chaplin*

- 28, no. 6, pp. 28 – 37, Dec. 2008.
- [10] H. K. Wimmer, “Monotonicity and Maximality of Solutions of Discrete-Time Algebraic Riccati Equations”, *Journal of Mathematics, Systems, Estimation and Control*, vol. 2, no. 2, pp. 219 – 235, 1992
  - [11] H. K. Wimmer and M. Pavon, “A comparison theorem for matrix Riccati difference equations”, *Systems & Control Letters*, vol. 19, pp. 233 – 239, 1992.
  - [12] L. E. Zachrisson, “On Optimal Smoothing of Continuous Time Kalman Processes”, *Information Sciences*, vol. 1, pp. 143 – 172, 1969.
  - [13] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York, 1971.
  - [14] A. Asif, “Fast Rauch-Tung-Striebel Smoother-Base Image Restoration for Noncausal Images”, *IEEE Signal Processing Letters*, vol. 11, no. 3, pp. 371 – 374, Mar. 2004.
  - [15] T. Kailath, A. H. Sayed and B. Hassibi, *Linear Estimation*, Prentice-Hall, Inc., Upper Saddle River, New Jersey, 2000.
  - [16] R. A. Monzingo, “Discrete Optimal Linear Smoothing for Systems with Uncertain Observations”, *IEEE Transactions on Information Theory*, vol. 21, no. 3, pp. 271 – 275, May 1975.



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## 8. Parameter Estimation

### 8.1 Introduction

Predictors, filters and smoothers have previously been described for state recovery under the assumption that the parameters of the generating models are correct. More often than not, the problem parameters are unknown and need to be identified. This section describes some standard statistical techniques for parameter estimation. Paradoxically, the discussed parameter estimation methods rely on having complete state information available. Although this is akin to a chicken-and-egg argument (state availability obviates the need for filters along with their attendant requirements for identified models), the task is not insurmountable.

The role of solution designers is to provide a cost benefit. That is, their objectives are to deliver improved performance at an acceptable cost. Inevitably, this requires simplifications so that the problems become sufficiently tractable and amenable to feasible solution. For example, suppose that speech emanating from a radio is too noisy and barely intelligible. In principle, high-order models could be proposed to equalise the communication channel, demodulate the baseband signal and recover the phonemes. Typically, low-order solutions tend to offer better performance because of the difficulty in identifying large numbers of parameters under low-SNR conditions. Consider also the problem of monitoring the output of a gas sensor and triggering alarms when environmental conditions become hazardous. Complex models could be constructed to take into account diurnal pressure variations, local weather influences and transients due to passing vehicles. It often turns out that low-order solutions exhibit lower false alarm rates because there are fewer assumptions susceptible to error. Thus, the absence of complete information need not inhibit solution development. Simple schemes may suffice, such as conducting trials with candidate parameter values and assessing the consequent error performance.

In maximum-likelihood estimation [1] – [5], unknown parameters  $\theta_1, \theta_2, \dots, \theta_M$ , are identified given states,  $x_k$ , by maximising a log-likelihood function,  $\log f(\theta_1, \theta_2, \dots, \theta_M | x_k)$ . For example, the subject of noise variance estimation was studied by Mehra in [6], where maximum-likelihood estimates (MLEs) were updated using

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“The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work” *John Von Neuman*

the Newton-Raphson method. Rife and Boorstyn obtained Cramér-Rao bounds for some MLEs, which “indicate the best estimation that can be made with the available data” [7]. Nayak *et al* used the pseudo-inverse to estimate unknown parameters in [8]. Belangér subsequently employed a least-squares approach to estimate the process noise and measurement noise variances [9]. A recursive technique for least-squares parameter estimation was developed by Strejc [10]. Dempster, Laird and Rubin [11] proved the convergence of a general purpose technique for solving joint state and parameter estimation problems, which they called the expectation-maximisation (EM) algorithm. They addressed problems where complete (state) information is not available to calculate the log-likelihood and instead maximised the expectation of  $\log f(\theta_1, \theta_2, \dots, \theta_M | z_k)$ , given incomplete measurements,  $z_k$ . That is, by virtue of Jensen’s inequality the unknowns are found by using an objective function (which are also called an approximate log-likelihood function),  $E\{\log f(\theta_1, \theta_2, \dots, \theta_M | z_k)\}$ , as a surrogate for  $\log f(\theta_1, \theta_2, \dots, \theta_M | x_k)$ .

The system identification literature is vast and some mature techniques have evolved. Subspace identification methods have been developed for general problems where a system’s stochastic inputs, deterministic inputs and outputs are available. The subspace algorithms [12] – [14] consist of two steps. First, the order of the system is identified from stacked vectors of the inputs and outputs. Then the unknown parameters are determined from an extended observability matrix.

Continuous-time maximum-likelihood estimation has been mentioned previously. Here, the attention is focussed on the specific problem of joint state and parameter estimation exclusively from discrete measurements of a system’s outputs. The developments proceed as follows. Section 8.2 reviews the maximum-likelihood estimation method for obtaining unknown parameters. The same estimates can be found using the method of least squares, which was pioneered by Gauss for fitting astronomical observations. Well known (filtering) EM algorithms for variance and state matrix estimation are described in Section 8.3. Improved parameter estimation accuracy can be obtained via smoothing EM algorithms, which are introduced in Section 8.4.

Although EM algorithms can yield improved state matrix and input process covariance estimates, it has been found that they are only accurate when the measurement noise is negligible. Similarly in subspace identification, least-squares estimation of unknown state-space matrices can lead to biased results when the states are corrupted by noise. This arises because the standard least-squares and maximum-likelihood estimates of a state matrix and an input process covariance *themselves* are biased in the presence of measurement noise. Therefore, a correction term is introduced in Section 8.5 to eliminate the bias error. This yields unbiased, consistent, closed-form estimates of a state matrix, an input

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“A hen is only an egg’s way of making another egg.” *Samuel Butler*

process covariance and a measurement noise covariance. Under simplifying conditions they are equal to MLEs and attain the corresponding Cramer-Rao Lower Bounds (CRLBs).

The use of the MLEs, filtering and smoothing EM algorithms discussed herein require caution. When perfect information and sufficiently large sample sizes are available, the corresponding likelihood functions are exact. However, the use of imperfect information leads to approximate likelihood functions and biased MLEs, which can degrade parameter estimation accuracy and follow-on filter/smoothing performance.

## 8.2 Maximum-Likelihood Estimation

### 8.2.1 General Method

Let  $p(\theta|x_k)$  denote the probability density function of an unknown parameter  $\theta$ , given samples of a discrete random variable  $x_k$ . An estimate,  $\hat{\theta}$ , can be obtained by finding the argument  $\theta$  that maximises the probability density function, that is,

$$\hat{\theta} = \arg \max_{\theta} p(\theta | x_k). \quad (1)$$

A solution can be found by setting  $\frac{\partial p(\theta | x_k)}{\partial \theta} = 0$  and solving for the unknown  $\theta$ .

Since the logarithm function is monotonic, a solution may be found equivalently by maximising

$$\hat{\theta} = \arg \max_{\theta} \log p(\theta | x_k) \quad (2)$$

and setting  $\frac{\partial \log p(\theta | x_k)}{\partial \theta} = 0$ . For exponential families of distributions, the use of (2) considerably simplifies the equations to be solved.

Suppose that  $N$  mutually independent samples of  $x_k$  are available, then the joint density function of all the observations is the product of the densities

$$\begin{aligned} f(\theta | x_k) &= p(\theta | x_1)p(\theta | x_2) \cdots p(\theta | x_N) \\ &= \prod_{k=1}^K p(\theta | x_k), \end{aligned} \quad (3)$$

which serves as a likelihood function. The MLE of  $\theta$  may be found maximising the log-likelihood

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“We are like eggs at present. And you cannot go on indefinitely being just an ordinary egg. We must hatch or go bad.” *Clive Staples Lewis*

$$\begin{aligned}
\hat{\theta} &= \arg \max_{\theta} \log f(\theta | x_k) \\
&= \arg \max_{\theta} \sum_{k=1}^N p(\theta | x_k)
\end{aligned} \tag{4}$$

by solving for a  $\theta$  that satisfies  $\frac{\partial \log f(\theta | x_k)}{\partial \theta} = \frac{\partial \log \sum_{k=1}^N p(\theta | x_k)}{\partial \theta} = 0$ . The above maximum-likelihood approach is applicable to a wide range of distributions. For example, the task of estimating the intensity of a Poisson distribution from measurements is demonstrated below.

**Example 1.** Suppose that  $N$  observations of integer  $x_k$  have a Poisson distribution  $f(x_k) = \frac{e^{-\mu} \mu^{x_k}}{x_k!}$ , where the intensity,  $\mu$ , is unknown. The corresponding log-likelihood function is

$$\begin{aligned}
\log f(\mu | x_k) &= \log \left( \frac{e^{-\mu} \mu^{x_1}}{x_1!} \frac{e^{-\mu} \mu^{x_2}}{x_2!} \frac{e^{-\mu} \mu^{x_3}}{x_3!} \dots \frac{e^{-\mu} \mu^{x_N}}{x_N!} \right) \\
&= \log \left( \frac{1}{x_1! x_2! \dots x_N!} \mu^{x_1 + x_2 + \dots + x_N} e^{-N\mu} \right) \\
&= -\log(x_1! x_2! \dots x_N!) + \log(\mu^{x_1 + x_2 + \dots + x_N}) - N\mu.
\end{aligned} \tag{5}$$

Taking  $\frac{\partial \log f(\mu | x_k)}{\partial \mu} = \frac{1}{\mu} \sum_{k=1}^N x_k - N = 0$  yields

$$\hat{\mu} = \frac{1}{N} \sum_{k=1}^N x_k. \tag{6}$$

Since  $\frac{\partial^2 \log f(\mu | x_k)}{\partial \mu^2} = -\frac{1}{\mu^2} \sum_{k=1}^N x_k$  is negative for all  $\mu$  and  $x_k \geq 0$ , the stationary point (6) occurs at a maximum of (5). That is to say,  $\hat{\mu}$  is indeed a maximum-likelihood estimate.

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“Therefore I would not have it unknown to Your Holiness, the only thing which induced me to look for another way of reckoning the movements of the heavenly bodies was that I knew that mathematicians by no means agree in their investigation thereof.” *Nicolaus Copernicus*“

## 8.2.2 State Matrix Estimation

From the Central Limit Theorem, which was mentioned in Chapter 6, the mean of a sufficiently large sample of independent identically distributed random variables will asymptotically approach a normal distribution. Consequently, in many maximum-likelihood estimation applications it is assumed that random variables are normally distributed. Recall that the normal (or Gaussian) probability density function of a discrete random variable  $x_k$  with mean  $\mu$  and covariance  $R_{xx}$  is

$$p(x_k) = \frac{1}{(2\pi)^{N/2} |R_{xx}|^{1/2}} \exp \left\{ -\frac{1}{2} (x_k - \mu)^T R_{xx}^{-1} (x_k - \mu) \right\}, \quad (7)$$

in which  $|R_{xx}|$  denotes the determinant of  $R_{xx}$ . A likelihood function for a sample of  $N$  independently identically distributed random variables is

$$f(x_k) = \prod_{k=1}^N p(x_k) = \frac{1}{(2\pi)^{N/2} |R_{xx}|^{N/2}} \sum_{k=1}^N \exp \left\{ -\frac{1}{2} (x_k - \mu)^T R_{xx}^{-1} (x_k - \mu) \right\}. \quad (8)$$

In general, it is more convenient to work with the log-likelihood function

$$\log f(x_k) = -\log (2\pi)^{N/2} |R_{xx}|^{N/2} - \frac{1}{2} \sum_{k=1}^N (x_k - \mu)^T R_{xx}^{-1} (x_k - \mu). \quad (9)$$

An example of estimating a model coefficient using the Gaussian log-likelihood approach is set out below.

**Example 2.** Consider an autoregressive order-one process  $x_{k+1} = a_0 x_k + w_k$ , with  $E\{w_k^2\} = \sigma_w^2$ , in which it is desired to estimate  $a_0 \in \mathbb{R}$  from samples of  $x_k$ . It follows from  $x_{k+1} \sim \mathcal{N}(a_0 x_k, \sigma_w^2)$  that

$$\log f(a_0 | x_{k+1}) = -\log (2\pi)^{N/2} \sigma_w^N - \frac{1}{2} \sum_{k=1}^N \sigma_w^{-2} (x_{k+1} - a_0 x_k)^2.$$

Setting  $\frac{\partial \log f(a_0 | x_{k+1})}{\partial a_0}$  equal to zero gives  $\sum_{k=1}^N x_{k+1} x_k = a_0 \sum_{k=1}^N x_k^2$  which results in the MLE

$$\hat{a}_0 = \frac{\sum_{k=1}^N x_{k+1} x_k}{\sum_{k=1}^N x_k^2}.$$

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“How wonderful that we have met with a paradox. Now we have some hope of making progress.”  
Niels Henrik David Bohr

Often within filtering and smoothing applications there are multiple parameters to be identified. Denote the unknown parameters by  $\theta_1, \theta_2, \dots, \theta_M$ , then the MLEs may be found by solving the  $M$  equations

$$\begin{aligned}\frac{\partial \log f(\theta_1, \theta_2, \dots, \theta_M | x_k)}{\partial \theta_1} &= 0 \\ \frac{\partial \log f(\theta_1, \theta_2, \dots, \theta_M | x_k)}{\partial \theta_2} &= 0 \\ &\vdots \\ \frac{\partial \log f(\theta_1, \theta_2, \dots, \theta_M | x_k)}{\partial \theta_M} &= 0.\end{aligned}$$

A vector parameter estimation example is outlined below.

**Example 3.** Consider the third-order autoregressive model

$$x_{k+3} + a_2 x_{k+2} + a_1 x_{k+1} + a_0 x_k = w_k \quad (10)$$

which can be written in the state-space form

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \end{bmatrix} = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix} + \begin{bmatrix} w_k \\ 0 \\ 0 \end{bmatrix}. \quad (11)$$

Assuming  $x_{1,k+1} \sim \mathcal{N}(-a_2 x_{1,k} - a_1 x_{2,k} - a_0 x_{3,k}, \sigma_w^2)$  and setting to zero the partial derivatives of the corresponding log-likelihood function with respect to  $a_0, a_1$  and  $a_2$  yields

$$-\begin{bmatrix} \sum_{k=1}^N x_{3,k}^2 & \sum_{k=1}^N x_{2,k} x_{3,k} & \sum_{k=1}^N x_{1,k} x_{3,k} \\ \sum_{k=1}^N x_{2,k} x_{3,k} & \sum_{k=1}^N x_{2,k}^2 & \sum_{k=1}^N x_{2,k} x_{1,k} \\ \sum_{k=1}^N x_{1,k} x_{3,k} & \sum_{k=1}^N x_{2,k} x_{1,k} & \sum_{k=1}^N x_{1,k}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^N x_{1,k+1} x_{3,k} \\ \sum_{k=1}^N x_{1,k+1} x_{2,k} \\ \sum_{k=1}^N x_{1,k+1} x_{1,k} \end{bmatrix}. \quad (12)$$

Hence, the MLEs are given by

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“If we all worked on the assumption that what is accepted as true is really true, there would be little hope of advance.” *Orville Wright*

$$\begin{bmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = - \begin{bmatrix} \sum_{k=1}^N x_{3,k}^2 & \sum_{k=1}^N x_{2,k} x_{3,k} & \sum_{k=1}^N x_{1,k} x_{3,k} \\ \sum_{k=1}^N x_{2,k} x_{3,k} & \sum_{k=1}^N x_{2,k}^2 & \sum_{k=1}^N x_{2,k} x_{1,k} \\ \sum_{k=1}^N x_{1,k} x_{3,k} & \sum_{k=1}^N x_{2,k} x_{1,k} & \sum_{k=1}^N x_{1,k}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{k=1}^N x_{1,k+1} x_{3,k} \\ \sum_{k=1}^N x_{1,k+1} x_{2,k} \\ \sum_{k=1}^N x_{1,k+1} x_{1,k} \end{bmatrix}. \quad (13)$$

### 8.2.3 Variance Estimation

MLEs can be similarly calculated for unknown variances, as is demonstrated by the following example.

**Example 4.** Consider a random variable generated by  $x_k = \mu + w_k$  where  $\mu \in \mathbb{R}$  is fixed and  $w_k \in \mathbb{R}$  is assumed to be a zero-mean Gaussian white input sequence. Since  $x_k \sim \mathcal{N}(\mu, \sigma_w^2)$ , it follows that

$$\log f(\sigma_w^2 | x_k) = -\frac{N}{2} \log 2\pi - \frac{N}{2} \log \sigma_w^2 - \frac{1}{2} \sigma_w^{-2} \sum_{k=1}^N (x_k - \mu)^2$$

and

$$\frac{\partial \log f(\sigma_w^2 | x_k)}{\partial \sigma_w^2} = -\frac{N}{2} (\sigma_w^2)^{-1} + \frac{1}{2} (\sigma_w^2)^{-2} \sum_{k=1}^N (x_k - \mu)^2.$$

From the solution of  $\frac{\partial \log f(\sigma_w^2 | x_k)}{\partial \sigma_w^2} = 0$ , the MLE is

$$\hat{\sigma}_w^2 = \frac{1}{N} \sum_{k=1}^N (x_k - \mu)^2, \text{ without replacement.} \quad (14)$$

If the random samples are taken from a population without replacement, the samples are not independent, the covariance between two different samples is nonzero and the MLE (14) is biased. If the sampling is done with replacement then the sample values are independent and the following correction applies

$$\hat{\sigma}_w^2 = \frac{1}{N-1} \sum_{k=1}^N (x_k - \mu)^2, \text{ with replacement.} \quad (15)$$

The corrected denominator within the above sample variance is only noticeable for small sample sizes, as the difference between (14) and (15) is negligible for

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“In science one tries to tell people, in such a way as to be understood by everyone, something that no-one knew before. But in poetry, it’s the exact opposite.” *Paul Adrien Maurice Dirac*



large  $N$ . The MLE (15) is unbiased, that is, its expected value equals the variance of the population. To confirm this property, observe that

$$\begin{aligned}
 E\{\sigma_w^2\} &= E\left\{\frac{1}{N-1} \sum_{k=1}^N (x_k - \mu)^2\right\} \\
 &= E\left\{\frac{1}{N-1} \sum_{k=1}^N x_k^2 - 2\mu x_k + \mu^2\right\} \\
 &= \frac{N}{N-1} (E\{x_k^2\} - E\{\mu^2\}). \tag{16}
 \end{aligned}$$

Using  $E\{x_k^2\} = \sigma_w^2 + \bar{x}^2$ ,  $E\{E\{x_k^2\}\} = E\{\sigma_w^2\} + E\{\mu^2\}$ ,  $E\{E\{x_k^2\}\} = E\{\mu^2\} = \mu^2$  and  $E\{\sigma_w^2\} = \sigma_w^2 / N$  within (16) yields  $E\{\sigma_w^2\} = \sigma_w^2$  as required. Unless stated otherwise, it is assumed herein that the sample size is sufficiently large so that  $N^{-1} \approx (N - 1)^{-1}$  and (15) may be approximated by (14). A caution about modelling error contributing bias is mentioned below.

**Example 5.** Suppose that the states considered in Example 4 are actually generated by  $x_k = \mu + w_k + s_k$ , where  $s_k$  is an independent input that accounts for the presence of modelling error. In this case, the assumption  $x_k \sim \mathcal{N}(\mu, \sigma_w^2 + \sigma_s^2)$  leads to  $\hat{\sigma}_w^2 + \hat{\sigma}_s^2 = \frac{1}{N} \sum_{k=1}^N (x_k - \mu)^2$ , in which case (14) is no longer an unbiased estimate of  $\sigma_w^2$ .

## 8.2.4 Cramér-Rao Lower Bound

The Cramér-Rao Lower Bound (CRLB) establishes a limit of precision that can be achieved for any unbiased estimate of a parameter  $\theta$ . It actually defines a lower bound for the variance  $\sigma_\theta^2$  of  $\hat{\theta}$ . As is pointed out in [1], since  $\hat{\theta}$  is assumed to be unbiased, the variance  $\sigma_\theta^2$  equals the parameter error variance. Determining lower bounds for parameter error variances is useful for model selection. Another way of selecting models involves comparing residual error variances [23]. A lucid introduction to Gaussian CRLBs is presented in [2]. An extensive survey that refers to the pioneering contributions of Fisher, Cramér and Rao appears in [4].

The bounds on the parameter variances are found from the inverse of the so-called Fisher information. A formal definition of the CRLB for scalar parameters is as follows.

**Theorem 1 (Cramér-Rao Lower Bound) [2] - [5]:** Assume that  $f(\theta | x_k)$  satisfies the following regularity conditions:

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“Everyone hears only what he understands.” *Johann Wolfgang von Goethe*

- (i)  $\frac{\partial \log f(\theta | x_k)}{\partial \theta}$  and  $\frac{\partial^2 \log f(\theta | x_k)}{\partial \theta^2}$  exist for all  $\theta$ , and
- (ii)  $E \left\{ \frac{\partial \log f(\theta | x_k)}{\partial \theta} \right\} = 0$ , for all  $\theta$ .

Define the Fisher information by

$$F(\theta) = -E \left\{ \frac{\partial^2 \log f(\theta | x_k)}{\partial \theta^2} \right\}, \quad (17)$$

where the derivative is evaluated at the actual value of  $\theta$ . Then the variance  $\sigma_{\hat{\theta}}^2$  of an unbiased estimate  $\hat{\theta}$  satisfies

$$\sigma_{\hat{\theta}}^2 \geq F^{-1}(\theta). \quad (18)$$

Proofs for the above theorem appear in [2] – [5].

**Example 6.** Suppose that samples of  $x_k = \mu + w_k$  are available, where  $w_k$  is a zero-mean Gaussian white input sequence and  $\mu \in \mathbb{R}$  is unknown. Since  $w_k \sim \mathcal{N}(0, \sigma_w^2)$ ,

$$\log f(\mu | x_k) = -\frac{N}{2} \log 2\pi - \frac{N}{2} \log \sigma_w^2 - \frac{1}{2} \sigma_w^{-2} \sum_{k=1}^N (x_k - \mu)^2$$

and

$$\frac{\partial \log f(\mu | x_k)}{\partial \mu} = \sigma_w^{-2} \sum_{k=1}^N (x_k - \mu).$$

Setting  $\frac{\partial \log f(\mu | x_k)}{\partial \mu} = 0$  yields the MLE<sup>9</sup>

$$\hat{\mu} = \frac{1}{N} \sum_{k=1}^N x_k, \quad (19)$$

which is unbiased because  $E\{\hat{\mu}\} = E\left\{\frac{1}{N} \sum_{k=1}^N x_k\right\} = E\left\{\frac{1}{N} \sum_{k=1}^N \mu\right\} = \mu$ . From

Theorem 1, the Fisher information is

$$F(\mu) = E \left\{ -\frac{\partial^2 \log f(\mu | x_k)}{\partial \mu^2} \right\} = E\{N\sigma_w^{-2}\} = N\sigma_w^{-2}$$

---

“Everyone has talent. What is rare is the courage to follow the talent to the dark place where it leads.”  
Erica Jong

and therefore

$$\sigma_{\hat{\mu}}^2 \geq \sigma_w^2 / N. \quad (20)$$

The above inequality suggests that a minimum of one sample is sufficient to bound the variance of the MLE (19). It is also apparent from (20) that the error variance of  $\mu$  decreases with increasing sample size.

The CRLB is extended for estimating a vector of parameters  $\theta_1, \theta_2, \dots, \theta_M$  by defining the  $M \times M$  Fisher information matrix

$$F_{ij}(\theta) = -E \left\{ \frac{\partial^2 \log f(\theta | x_k)}{\partial \theta_i \partial \theta_j} \right\}, \quad (21)$$

for  $i, j = 1 \dots M$ . The parameter error variances are then bounded by the diagonal elements of Fisher information matrix inverse

$$\sigma_{\hat{\theta}_i}^2 \geq F_{ii}^{-1}(\theta). \quad (22)$$

Formal vector CRLB theorems and accompanying proofs are detailed in [2] – [5].

**Example 7.** Consider the problem of estimating both  $\mu$  and  $\sigma_w^2$  from  $N$  samples of  $x_k = \mu + w_k$ , with  $w_k \sim \mathcal{N}(0, \sigma_w^2)$ . Recall from Example 6 that

$$\log f(\mu, \sigma_w^2 | x_k) = -\frac{N}{2} \log 2\pi - \frac{N}{2} \log \sigma_w^2 - \frac{1}{2} \sigma_w^{-2} \sum_{k=1}^N (x_k - \mu)^2.$$

Therefore,  $\frac{\partial \log f(\mu, \sigma_w^2 | x_k)}{\partial \mu} = \sigma_w^{-2} \sum_{k=1}^N (x_k - \mu)$  and  $\frac{\partial^2 \log f(\mu, \sigma_w^2 | x_k)}{\partial \mu^2} = -N\sigma_w^{-2}$ . In Example 4 it is found that  $\frac{\partial \log f(\mu, \sigma_w^2 | x_k)}{\partial \sigma_w^2} = -\frac{N}{2}(\sigma_w^2)^{-1} + \frac{1}{2}(\sigma_w^2)^{-2} \sum_{k=1}^N (x_k - \mu)^2$ , which implies

$$\begin{aligned} \frac{\partial^2 \log f(\mu, \sigma_w^2 | x_k)}{\partial (\sigma_w^2)^2} &= \frac{N}{2} (\sigma_w^2)^{-2} - (\sigma_w^2)^{-3} \sum_{k=1}^N (x_k - \mu)^2 \\ &= \frac{N}{2} (\sigma_w^2)^{-2} - N(\sigma_w^2)^{-2} \end{aligned}$$

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“Laying in bed this morning contemplating how amazing it would be if somehow Oscar Wilde and Mae West could twitter from the grave.” @DitaVonTeese

$$= -\frac{N}{2} \sigma_w^{-4},$$

$$\frac{\partial^2 \log f(\mu, \sigma_w^2 | x_k)}{\partial \mu \partial \sigma_w^2} = -(\sigma_w^2)^{-2} \sum_{k=1}^N (x_k - \mu) \text{ and } E \left\{ \frac{\partial^2 \log f(\mu, \sigma_w^2 | x_k)}{\partial \mu \partial \sigma_w^2} \right\} = 0.$$

The Fisher information matrix and its inverse are then obtained from (21) as

$$F(u, \sigma_w^2) = \begin{bmatrix} N\sigma_w^{-2} & 0 \\ 0 & 0.5N\sigma_w^{-4} \end{bmatrix}, \quad F^{-1}(u, \sigma_w^2) = \begin{bmatrix} \sigma_w^2 / N & 0 \\ 0 & 2\sigma_w^4 / N \end{bmatrix}.$$

It is found from (22) that the lower bounds for the MLE variances are  $\sigma_\mu^2 \geq \sigma_w^2 / N$  and  $\sigma_{\hat{\sigma}_w^2}^2 \geq 2\sigma_w^4 / N$ . The impact of modelling error on parameter estimation accuracy is examined below.

**Example 8.** Consider the problem of estimating  $\sigma_w^2$  given samples of states which are generated by  $x_k = \mu + w_k + s_k$ , where  $s_k$  is an independent sequence that accounts for the presence of modelling error. From the assumption  $x_k \sim \mathcal{N}(\mu, \sigma_w^2 + \sigma_s^2)$ , the associated log likelihood function is

$$\frac{\partial \log f(\sigma_w^2 | x_k)}{\partial \sigma_w^2} = -\frac{N}{2} (\sigma_w^2 + \sigma_s^2)^{-1} + \frac{1}{2} (\sigma_w^2 + \sigma_s^2)^{-2} \sum_{k=1}^N (x_k - \mu)^2,$$

which leads to  $\frac{\partial^2 \log f(\sigma_w^2 | x_k)}{\partial (\sigma_w^2)^2} = -\frac{N}{2} (\sigma_w^2 + \sigma_s^2)^{-2}$ , that is,  $\sigma_{\hat{\sigma}_w^2}^2 \geq$

$2(\sigma_w^2 + \sigma_s^2)^2 / N$ . Thus, parameter estimation accuracy degrades as the variance of the modelling error increases. If  $\sigma_s^2$  is available *a priori* then setting  $\frac{\partial \log f(\sigma_w^2 | x_k)}{\partial \sigma_w^2} = 0$  leads to the improved estimate

$$\hat{\sigma}_w^2 = -\sigma_s^2 + \frac{1}{N} \sum_{k=1}^K (x_k - \mu)^2.$$

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“There are only two kinds of people who are really fascinating; people who know absolutely everything, and people who know absolutely nothing.” *Oscar Fingal O’Flahertie Wills Wilde*

## 8.3 Filtering EM Algorithms

### 8.3.1 Background

The EM algorithm [3], [7], [11], [15] – [17], [19] – [22] is a general purpose technique for solving joint state and parameter estimation problems. In maximum-likelihood estimation, it is desired to estimate parameters  $\theta_1, \theta_2, \dots, \theta_M$ , given states by maximising the log-likelihood  $\log f(\theta_1, \theta_2, \dots, \theta_M | x_k)$ . When complete state information is not available to calculate the log-likelihood, the expectation of  $\log f(\theta_1, \theta_2, \dots, \theta_M | x_k)$ , given incomplete measurements,  $z_k$ , is maximised instead. This basic technique was in use prior to Dempster, Laird and Rubin naming it the EM algorithm 1977 [11]. They published a general formulation of the algorithm, which consists of iterating an expectation step and a maximisation step. Their expectation step involves least squares calculations on the incomplete observations using the current parameter iterations to estimate the underlying states. In the maximisation step, the unknown parameters are re-estimated by maximising a joint log likelihood function using state estimates from the previous expectation step. This sequence is repeated for either a finite number of iterations or until the estimates and the log likelihood function are stable. Dempster, Laird and Rubin [11] also established parameter map conditions for the convergence of the algorithm, namely that the incomplete data log likelihood function is monotonically nonincreasing.

Wu [16] subsequently noted an equivalence between the conditions for a map to be closed and the continuity of a function. In particular, if the likelihood function satisfies certain modality, continuity and differentiability conditions, the parameter sequence converges to some stationary value. A detailed analysis of Wu's convergence results appears in [3]. Shumway and Stoffer [15] introduced a framework that is employed herein, namely, the use of a Kalman filter within the expectation step to recover the states. Feder and Weinstein [17] showed how a multiparameter estimation problem can be decoupled into separate maximum likelihood estimations within an EM algorithm. Some results on the convergence of EM algorithms for variance and state matrix estimation [19] – [20] are included within the developments below.

### 8.3.2 Measurement Noise Variance Estimation

#### 8.3.2.1 EM Algorithm

The problem of estimating parameters from incomplete information has been previously studied in [11] – [16]. It is noted in [11] that the likelihood functions for variance estimation do not exist in explicit closed form. This precludes

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“I’m no model lady. A model is just an imitation of the real thing.” *Mary (Mae) Jane West*

straightforward calculation of the Hessians required in [3] to assert convergence. Therefore, an alternative analysis is presented here to establish the monotonicity of variance iterations.

The expectation step described below employs the approach introduced in [15] and involves the use of a Kalman filter to obtain state estimates. The maximisation step requires the calculation of decoupled MLEs similarly to [17]. Measurements of a linear time-invariant system are modelled by

$$x_{k+1} = Ax_k + Bw_k, \quad (23)$$

$$y_k = Cx_k + Dw_k, \quad (24)$$

$$z_k = y_k + v_k, \quad (25)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$  and  $w_k$ ,  $v_k$  are stationary processes with  $E\{w_k\} = 0$ ,  $E\{w_j w_k^T\} = Q\delta_{jk}$ ,  $E\{v_k\} = 0$  and  $E\{v_j v_k^T\} = R\delta_{jk}$ . To simplify the presentation, it is initially assumed that the direct feed-through matrix,  $D$ , is zero. A nonzero  $D$  will be considered later.

Suppose that it is desired to estimate  $R = \text{diag}(\sigma_{1,v}^2, \sigma_{2,v}^2, \dots, \sigma_{p,v}^2)$  given  $N$  samples of  $z_k$  and  $y_k$ . Let  $z_{i,k}$ ,  $y_{i,k}$  and  $v_{i,k}$  denote the  $i^{\text{th}}$  element of the vectors  $z_k$ ,  $y_k$  and  $v_k$ , respectively. Then (25) may be written in terms of its  $i$  components,  $z_{i,k} = y_{i,k} + v_{i,k}$ , that is,

$$v_{i,k} = z_{i,k} - y_{i,k}. \quad (26)$$

From the assumption  $v_{i,k} \sim \mathcal{N}(0, \sigma_{i,v}^2)$ , an MLE for the unknown  $\sigma_{i,v}^2$  is obtained from the sample variance formula

$$\hat{\sigma}_{i,v}^2 = \frac{1}{N} \sum_{k=1}^N (z_{i,k} - y_{i,k})^2. \quad (27)$$

An EM algorithm for updating the measurement noise variance estimates is described as follows. Assume that there exists an estimate  $\hat{R}^{(u)} = \text{diag}((\hat{\sigma}_{1,v}^{(u)})^2, (\hat{\sigma}_{2,v}^{(u)})^2, \dots, (\hat{\sigma}_{p,v}^{(u)})^2)$  of  $R$  at iteration  $u$ . A Kalman filter designed with  $\hat{R}^{(u)}$  may then be employed to produce corrected output estimates  $\hat{y}_{k/k}^{(u)}$ . The filter's design Riccati equation is given by

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“There are known knowns. These are things we know that we know. There are known unknowns. That is to say, there are things that we know we don't know. But there are also unknown unknowns. There are things we don't know we don't know.” *Donald Henry Rumsfeld*

$$P_{k+1/k}^{(u)} = (A - K_k^{(u)}C)P_{k/k-1}^{(u)}(A - K_k^{(u)}C)^T + K_k^{(u)}\hat{R}^{(u)}(K_k^{(u)})^T + BQB^T, \quad (28)$$

where  $K_k^{(u)} = AP_{k/k-1}^{(u)}C^T(CP_{k/k-1}^{(u)}C^T + \hat{R}^{(u)})^{-1}$  is the predictor gain. The output estimates are calculated from

$$\begin{bmatrix} \hat{x}_{k+1/k}^{(u)} \\ \hat{x}_{k/k}^{(u)} \end{bmatrix} = \begin{bmatrix} (A - K_k^{(u)}C) & K_k^{(u)} \\ (I - L_k^{(u)}C) & L_k^{(u)} \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1}^{(u)} \\ z_k \end{bmatrix}, \quad (29)$$

$$\hat{y}_{k/k}^{(u)} = C\hat{x}_{k/k}^{(u)}, \quad (30)$$

where  $L_k^{(u)} = P_{k/k-1}^{(u)}C^T(CP_{k/k-1}^{(u)}C^T + \hat{R}^{(u)})^{-1}$  is the filter gain.

**Procedure 1 [19].** Assume that an initial estimate  $\hat{R}^{(1)}$  of  $R$  is available. Subsequent estimates,  $\hat{R}^{(u)}$ ,  $u > 1$ , are calculated by repeating the following two-step procedure.

Step 1. Operate the Kalman filter (29) – (30) designed with  $\hat{R}^{(u)}$  to obtain corrected output estimates  $\hat{y}_{k/k}^{(u)}$ .

Step 2. For  $i = 1, \dots, p$ , use  $\hat{y}_{k/k}^{(u)}$  instead of  $y_k$  within (27) to obtain  $\hat{R}^{(u+1)} = \text{diag}((\hat{\sigma}_{1,v}^{(u+1)})^2, (\hat{\sigma}_{2,v}^{(u+1)})^2, \dots, (\hat{\sigma}_{p,v}^{(u+1)})^2)$ .

### 8.3.2.2 Properties

The above EM algorithm involves a repetition of two steps: the states are deduced using the current variance estimates and then the variances are re-identified from the latest states. Consequently, a two-part argument is employed to establish the monotonicity of the variance sequence. For the expectation step, it is shown that monotonically non-increasing variance iterates lead to monotonically non-increasing error covariances. Then for the maximisation step, it is argued that monotonic error covariances result in a monotonic measurement noise variance sequence. The design Riccati difference equation (28) can be written as

$$P_{k+1/k}^{(u)} = (A - K_k^{(u)}C)P_{k/k-1}^{(u)}(A - K_k^{(u)}C)^T + K_k^{(u)}R(K_k^{(u)})^T + Q + S_k^{(u)}, \quad (31)$$

where  $S_k^{(u)} = K_k^{(u)}(\hat{R}^{(u)} - R)(K_k^{(u)})^T$  accounts for the presence of parameter error.

Subtracting  $x_k$  from  $\hat{x}_{k/k}^{(u)}$  yields

$$\tilde{x}_{k/k}^{(u)} = (I - L_k^{(u)}C)\tilde{x}_{k/k-1}^{(u)} - L_k^{(u)}v_k, \quad (32)$$

---

“I want minimum information given with maximum politeness.” *Jacqueline (Jackie) Lee Bouvier Kennedy Onassis*

where  $\tilde{x}_{k/k}^{(u)} = x_k - \hat{x}_{k/k}^{(u)}$  and  $\tilde{x}_{k/k-1}^{(u)} = x_k - \hat{x}_{k/k-1}^{(u)}$  are the corrected and predicted state errors, respectively. The observed corrected error covariance is defined as  $\Sigma_{k/k}^{(u)} = E\{\tilde{x}_{k/k}^{(u)}(\tilde{x}_{k/k}^{(u)})^T\}$  and obtained from

$$\begin{aligned}\Sigma_{k/k}^{(u)} &= (I - L_k^{(u)}C)\Sigma_{k/k-1}^{(u)}(I - L_k^{(u)}C)^T + L_k^{(u)}R(L_k^{(u)})^T \\ &= \Sigma_{k/k-1}^{(u)} - \Sigma_{k/k-1}^{(u)}C^T(C\Sigma_{k/k-1}^{(u)}C^T + R)^{-1}C\Sigma_{k/k-1}^{(u)},\end{aligned}\quad (33)$$

where  $\Sigma_{k/k-1}^{(u)} = E\{\tilde{x}_{k/k-1}^{(u)}(\tilde{x}_{k/k-1}^{(u)})^T\}$ . The observed predicted state error satisfies

$$\tilde{x}_{k+1/k}^{(u)} = A\tilde{x}_{k/k}^{(u)} + Bw_k. \quad (34)$$

Hence, the observed predicted error covariance obeys the recursion

$$\Sigma_{k+1/k}^{(u)} = A\Sigma_{k/k}^{(u)}A^T + BQB^T. \quad (35)$$

Some observations concerning the above error covariances are described below. These results are used subsequently to establish the monotonicity of the above EM algorithm.

**Lemma 1 [19]:** *In respect of Procedure 1 for estimating  $R$ , suppose the following:*

- (i) *the data  $z_k$  has been generated by (23) – (25) in which  $A$ ,  $B$ ,  $C$ ,  $Q$  are known,  $|\lambda_i(A)| < 1$ ,  $i = 1, \dots, n$ , and the pair  $(A, C)$  is observable;*
- (ii) *there exist  $P_{1/0}^{(2)} \leq P_{1/0}^{(1)}$  and  $R \leq \hat{R}^{(2)} \leq \hat{R}^{(1)}$  (or  $P_{1/0}^{(1)} \leq P_{1/0}^{(2)}$  and  $\hat{R}^{(1)} \leq \hat{R}^{(2)} \leq R$ ).*

*Then:*

- (i)  $\Sigma_{k+1/k}^{(u)} \leq P_{k+1/k}^{(u)}$ ;
- (ii)  $\Sigma_{k/k}^{(u)} \leq P_{k/k}^{(u)}$ ;
- (iii)  $R \leq \hat{R}^{(u+1)} \leq \hat{R}^{(u)}$  implies  $P_{k+1/k}^{(u+1)} \leq P_{k+1/k}^{(u)}$  (or  $\hat{R}^{(u)} \leq \hat{R}^{(u+1)} \leq R$  implies  $P_{k+1/k}^{(u)} \leq P_{k+1/k}^{(u+1)}$ )

*for all  $u \geq 1$ .*

**Proof:**

- (i) *Condition (i) ensures that the problem is well-posed. Condition (ii) stipulates that  $S_k^{(1)} \geq 0$ , which is the initialisation step for an induction argument. For the inductive step, subtracting (33) from (31) yields  $P_{k+1/k}^{(u)}$*



- $-\Sigma_{k+1/k}^{(u)} = (A - K_k^{(u)}C)(P_{k/k-1}^{(u)} - \Sigma_{k/k-1}^{(u)})(A - K_k^{(u)}C)^T + S_k^{(u)}$  and thus  
 $\Sigma_{k/k-1}^{(u)} \leq P_{k/k-1}^{(u)}$  implies  $\Sigma_{k+1/k}^{(u)} \leq P_{k+1/k}^{(u)}$ .  
(ii) The result is immediate by considering  $A = I$  within the proof for (i).  
(iii) The condition  $\hat{R}^{(u+1)} \leq \hat{R}^{(u)}$  ensures that  

$$\begin{bmatrix} Q & A^T \\ A & -C^T(\hat{R}^{(u+1)})^{-1}C \end{bmatrix} \leq \begin{bmatrix} Q & A^T \\ A & -C^T(\hat{R}^{(u)})^{-1}C \end{bmatrix},$$
 which together with  
 $P_{1/0}^{(u+1)} \leq P_{1/0}^{(u)}$  within Theorem 2 of Chapter 7 results in  $P_{k+1/k}^{(u+1)} \leq P_{k+1/k}^{(u)}$ .

Thus the sequences of observed prediction and correction error covariances are bounded above by the design prediction and correction error covariances. Next, it is shown that the observed error covariances are monotonically non-increasing (or non-decreasing).

**Lemma 2 [19]:** Under the conditions of Lemma 1:

- i)  $\Sigma_{k+1/k}^{(u+1)} \leq \Sigma_{k+1/k}^{(u)}$  (or  $\Sigma_{k+1/k}^{(u)} \leq \Sigma_{k+1/k}^{(u+1)}$ ) and  
ii)  $\Sigma_{k/k}^{(u+1)} \leq \Sigma_{k/k}^{(u)}$  (or  $\Sigma_{k/k}^{(u)} \leq \Sigma_{k/k}^{(u+1)}$ ).

**Proof:** To establish that the solution of (33) is monotonic non-increasing, from Theorem 2 of Chapter 7, it is required to show that

$$\begin{bmatrix} Q + K_k^{(u+1)}R(K_k^{(u+1)})^T & (A - K_k^{(u+1)}C)^T \\ A - K_k^{(u+1)}C & 0 \end{bmatrix} \leq \begin{bmatrix} Q + K_k^{(u)}R(K_k^{(u)})^T & (A - K_k^{(u)}C)^T \\ A - K_k^{(u)}C & 0 \end{bmatrix}.$$

Since A, Q and R are time-invariant, it suffices to show that

$$\begin{bmatrix} L_k^{(u+1)}(L_k^{(u+1)})^T & (I - L_k^{(u+1)}C)^T \\ I - L_k^{(u+1)}C & 0 \end{bmatrix} \leq \begin{bmatrix} L_k^{(u)}(L_k^{(u)})^T & (I - L_k^{(u)}C)^T \\ I - L_k^{(u)}C & 0 \end{bmatrix}. \quad (36)$$

Note for an X and Y satisfying  $I \geq Y \geq X \geq 0$  that  $YY^T - XX^T \geq (I - X)(I - X)^T - (I - Y)(I - Y)^T$ . Therefore,  $\hat{R}^{(u+1)} \leq \hat{R}^{(u)}$  and  $P_{k+1/k}^{(u+1)} \leq P_{k+1/k}^{(u)}$  (from Lemma 1) imply  $L^{(u+1)}C \leq L^{(u)}C \leq I$  and thus (36) follows.

It is established below that monotonic non-increasing error covariances result in a monotonic non-increasing measurement noise variance sequence.

**Lemma 3 [19]:** In respect of Procedure 1 for estimating R, suppose the following:

- (i) the data  $z_k$  has been generated by (23) – (25) in which A, B, C, Q are known,  $|\lambda_i(A)| < 1$ ,  $i = 1, \dots, n$  and the pair (A, C) is observable;

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“The Internet is the world’s largest library. It’s just that all the books are on the floor.” John Allen Paulos

(ii) there exist  $\hat{R}^{(1)} \geq R \geq 0$  and  $P_{1/0}^{(u+1)} \leq P_{1/0}^{(u)}$  (or  $P_{1/0}^{(u)} \leq P_{1/0}^{(u+1)}$ ) for all  $u > 1$ .

Then  $\hat{R}^{(u+1)} \leq \hat{R}^{(u)}$  (or  $\hat{R}^{(u)} \leq \hat{R}^{(u+1)}$ ) for all  $u > 1$ .

**Proof:** Let  $C_i$  denote the  $i^{\text{th}}$  row of  $C$ . The approximate MLE within Procedure 1 is written as

$$(\hat{\sigma}_{i,v}^{(u+1)})^2 = \frac{1}{N} \sum_{k=1}^N (z_{i,k} - C_i \hat{x}_{k/k}^{(u)})^2 \quad (37)$$

$$= \frac{1}{N} \sum_{k=1}^N (C_i \tilde{x}_{k/k}^{(u)} + v_{i,k})^2 \quad (38)$$

$$= C_i \Sigma_{k/k}^{(u)} C_i^T + \sigma_{i,v}^2 \quad (39)$$

and thus  $\hat{R}^{(u+1)} = C \Sigma_{k/k}^{(u)} C^T + R$ . Since  $\hat{R}^{(u+1)}$  is affine to  $\Sigma_{k/k}^{(u)}$ , which from Lemma 2 is monotonically non-increasing, it follows that  $\hat{R}^{(u+1)} \leq \hat{R}^{(u)}$ .

If the estimation problem is dominated by measurement noise, the measurement noise MLEs converge to the actual values.

**Lemma 4 [19]:** Under the conditions of Lemma 3,

$$\lim_{Q \rightarrow 0, R^{-1} \rightarrow 0, u \rightarrow \infty} \hat{R}^{(u+1)} = R. \quad (40)$$

**Proof:** By inspection of  $L_k^{(u)} = P_{k/k-1}^{(u)} C^T (C P_{k/k-1}^{(u)} C^T + R^{(u)})^{-1}$ , it follows that  $\lim_{Q \rightarrow 0, R^{-1} \rightarrow 0, u \rightarrow \infty} L_k^{(u)} = 0$ . Therefore,  $\lim_{Q \rightarrow 0, R^{-1} \rightarrow 0, u \rightarrow \infty} \hat{x}_{k/k}^{(u)} = 0$  and  $\lim_{Q \rightarrow 0, R^{-1} \rightarrow 0} z_k = v_k$ , which implies (40), since the MLE (37) is unbiased for large  $N$ .

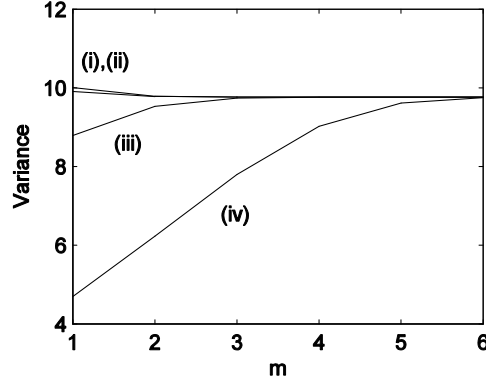


Fig. 1. Variance MLEs (27) versus iteration number for Example 9: (i) EM algorithm with  $(\hat{\sigma}_v^{(1)})^2 = 14$ , (ii) EM algorithm with  $(\hat{\sigma}_v^{(1)})^2 = 12$ , (iii) Newton-Raphson with  $(\hat{\sigma}_v^{(1)})^2 = 14$  and (iv) Newton-Raphson with  $(\hat{\sigma}_v^{(1)})^2 = 12$ .

**Example 9.** In respect of the problem (23) – (25), assume  $A = 0.9$ ,  $B = C = 1$  and  $\sigma_w^2 = 0.1$  are known. Suppose that  $\sigma_v^2 = 10$  but is unknown. Samples  $z_k$  and  $\hat{x}_{k/k}^{(u)}$  were generated from  $N = 20,000$  realisations of zero-mean Gaussian  $w_k$  and  $v_k$ . The sequence of MLEs obtained using Procedure 1, initialised with  $(\hat{\sigma}_v^{(1)})^2 = 14$  and 12 are indicated by traces (i) and (ii) of Fig. 1, respectively. The variance sequences are monotonically decreasing, which is consistent with Lemma 3. The figure shows that the MLEs converge (to a local maximum of the approximate log-likelihood function), and are reasonably close to the actual value of  $\sigma_v^2 = 10$ . This illustrates the high measurement noise observation described by Lemma 4. An alternative to the EM algorithm involves calculating MLEs using the Newton-Raphson method [5], [6]. The calculated Newton-Raphson measurement noise variance iterates, initialised with  $(\hat{\sigma}_v^{(1)})^2 = 14$  and 12 are indicated by traces (iii) and (iv) of Fig. 1, respectively. It can be seen that the Newton-Raphson estimates converge to those of the EM algorithm, albeit at a slower rate.

### 8.3.3 Process Noise Variance Estimation

#### 8.3.3.1 EM Algorithm

In respect of the model (23), suppose that it is desired to estimate  $Q$  given  $N$  samples of  $x_{k+1}$ . The vector states within (23) can be written in terms of its  $i$  components,  $x_{i,k+1} = A_i x_k + w_{i,k}$ , that is

$$w_{i,k} = A_i x_k - x_{i,k+1}, \quad (41)$$

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“Getting information off the internet is like taking a drink from a fire hydrant.” *Mitchell David Kapor*“

where  $w_{i,k} = B_i w_k$ ,  $A_i$  and  $B_i$  refer the  $i^{\text{th}}$  row of  $A$  and  $B$ , respectively. Assume that  $w_{i,k} \sim \mathcal{N}(0, \sigma_{i,w}^2)$ , where  $\sigma_{i,w}^2 \in \mathbb{R}$  is to be estimated. An MLE for the scalar  $\sigma_{i,w}^2 = B_i Q B_i^T$  can be calculated from the sample variance formula

$$\sigma_{i,w}^2 = \frac{1}{N} \sum_{k=1}^N w_{i,k} w_{i,k}^T \quad (42)$$

$$= \frac{1}{N} \sum_{k=1}^N (x_{i,k+1} - A_i x_k)(x_{i,k+1} - A_i x_k)^T \quad (43)$$

$$= \frac{1}{N} \sum_{k=1}^N B_i w_k w_k^T B_i^T \quad (44)$$

$$= B_i \left( \frac{1}{N} \sum_{k=1}^N w_k w_k^T \right) B_i^T. \quad (45)$$

Substituting  $w_k = Ax_k - x_{k+1}$  into (45) and noting that  $\sigma_{i,w}^2 = B_i Q B_i^T$  yields

$$\hat{Q} = \frac{1}{N} \sum_{k=1}^N (Ax_k - x_{k+1})(Ax_k - x_{k+1})^T, \quad (46)$$

which can be updated as follows.

**Procedure 2 [19].** Assume that an initial estimate  $\hat{Q}^{(1)}$  of  $Q$  is available. Subsequent estimates can be found by repeating the following two-step algorithm.

Step 1. Operate the filter recursions (29) designed with  $\hat{Q}^{(u)}$  on the measurements (25) over  $k \in [1, N]$  to obtain corrected state estimates

$$\hat{x}_{k/k}^{(u)} \text{ and } \hat{x}_{k+1/k+1}^{(u)}.$$

Step 2. For  $i = 1, \dots, n$ , use  $\hat{x}_{k/k}^{(u)}$  and  $\hat{x}_{k+1/k+1}^{(u)}$  instead of  $x_k$  and  $x_{k+1}$  within (46) to obtain  $\hat{Q}^{(u+1)} = \text{diag}((\hat{\sigma}_{1,w}^{(u+1)})^2, (\hat{\sigma}_{2,w}^{(u+1)})^2, \dots, (\hat{\sigma}_{n,w}^{(u+1)})^2)$ .

### 8.3.3.2 Properties

Similarly to Lemma 1, it can be shown that a monotonically non-increasing (or non-decreasing) sequence of process noise variance estimates results in a monotonically non-increasing (or non-decreasing) sequence of design and observed error covariances, see [19]. The converse case is stated below, namely, the sequence of variance iterates is monotonically non-increasing, provided the

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“Information on the Internet is subject to the same rules and regulations as conversations at a bar.”  
George David Lundberg

estimates and error covariances are initialised appropriately. The accompanying proof makes use of

$$\begin{aligned}\hat{\mathbf{x}}_{k+1/k+1}^{(u)} - A\hat{\mathbf{x}}_{k/k}^{(u)} &= \hat{\mathbf{x}}_{k+1/k}^{(u)} + L_{k+1}^{(u)}(z_{k+1} - C\hat{\mathbf{x}}_{k+1/k}^{(u)}) - A\hat{\mathbf{x}}_{k/k}^{(u)} \\ &= A\hat{\mathbf{x}}_{k/k}^{(u)} + L_{i,k+1}^{(u)}(z_{k+1} - C\hat{\mathbf{x}}_{k+1/k}^{(u)}) - A\hat{\mathbf{x}}_{k/k}^{(u)} \\ &= L_k^{(u)}(C\tilde{\mathbf{x}}_{k+1/k}^{(u)} + v_{k+1}).\end{aligned}\quad (47)$$

The components of (47) are written as

$$\hat{x}_{i,k+1/k+1}^{(u)} - a_i \hat{x}_{k/k}^{(u)} = L_{i,k+1}^{(u)}(C\tilde{x}_{k+1/k}^{(u)} + v_{k+1}), \quad (48)$$

where  $L_{i,k+1}^{(u)}$  is the  $i^{\text{th}}$  row of  $L_{k+1}^{(u)}$ .

**Lemma 5 [19]:** *In respect of Procedure 2 for estimating  $Q$ , suppose the following:*

(i) *the data  $z_k$  has been generated by (23) – (25) in which  $A, B, C, R$  are known,  $|\lambda_i(A)| < 1, i = 1, \dots, n$  and the pair  $(A, C)$  is observable;*

(ii) *there exist  $\hat{Q}^{(1)} \geq Q \geq 0$  and  $P_{1/0}^{(u+1)} \leq P_{1/0}^{(u)}$  (or  $P_{1/0}^{(u)} \leq P_{1/0}^{(u+1)}$ ) for all  $u > 1$ .*

*Then  $\hat{Q}^{(u+1)} \leq \hat{Q}^{(u)}$  (or  $\hat{Q}^{(u)} \leq \hat{Q}^{(u+1)}$ ) for all  $u > 1$ .*

**Proof:** Using (47) within (46) gives

$$\begin{aligned}(\hat{\sigma}_{i,w}^{(u)})^2 &= \frac{1}{N} L_{i,k+1}^{(u)} \left( \sum_{k=1}^N C\tilde{\mathbf{x}}_{k+1/k}^{(u)} + v_{k+1} \right)^2 (L_{i,k+1}^{(u)})^T \\ &= L_{i,k+1}^{(u)} (C\Sigma_{k+1/k}^{(u)} C^T + R) (L_{i,k+1}^{(u)})^T\end{aligned}\quad (49)$$

and thus  $\hat{Q}^{(u+1)} = L_{k+1}^{(u)} (C\Sigma_{k+1/k}^{(u)} C^T + R) (L_{k+1}^{(u)})^T$ . Since  $\hat{Q}^{(u+1)}$  varies with  $L_{k+1}^{(u)} (L_{j,k+1}^{(u)})^T$  and  $\Sigma_{k+1/k}^{(u)}$ , which from Lemma 2 are monotonically non-increasing, it follows that  $\hat{Q}^{(u+1)} \leq \hat{Q}^{(u)}$ .

It is observed that the approximate MLEs asymptotically approach the actual values when the SNR is sufficiently high.

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“I must confess that I’ve never trusted the Web. I’ve always seen it as a coward’s tool. Where does it live? How do you hold it personally responsible? Can you put a distributed network of fibre-optic cable on notice? And is it male or female? In other words, can I challenge it to a fight?” *Stephen Tyrone Colbert*

**Lemma 6 [19]:** Under the conditions of Lemma 5,

$$\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{Q}^{(u+1)} = Q. \quad (50)$$

**Proof:** It is straight forward to show that  $\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0} L_{i,k} C = I$  and therefore

$\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{x}_{k/k}^{(u)} = x_k$ , which implies (50), since the MLE (46) is unbiased for large  $N$ .

**Example 10.** For the model described in Example 8, suppose that  $\sigma_v^2 = 0.01$  is known, and  $(\hat{\sigma}_w^{(1)})^2 = 0.1$  but is unknown. Procedure 2 and the Newton-Raphson method [5], [6] were used to jointly estimate the states and the unknown variance. Some example variance iterations, initialised with  $(\hat{\sigma}_w^{(1)})^2 = 0.14$  and 0.12, are shown in Fig. 2. The EM algorithm estimates are seen to be monotonically decreasing, which is in agreement with Lemma 5. At the final iteration, the approximate MLEs do not quite reach the actual value of  $(\hat{\sigma}_w^{(1)})^2 = 0.1$ , because the presence of measurement noise results in imperfect state reconstruction and introduces a small bias (see Example 5). The figure also shows that MLEs calculated via the Newton-Raphson method converge at a slower rate.

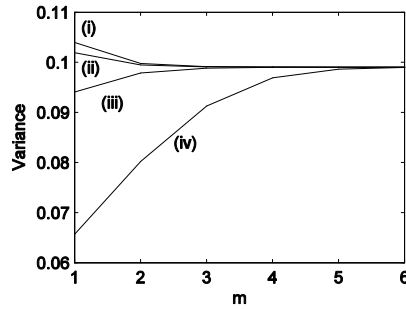


Fig. 2. Variance MLEs (46) versus iteration number for Example 10: (i) EM algorithm with  $(\hat{\sigma}_w^{(1)})^2 = 0.14$ , (ii) EM algorithm with  $(\hat{\sigma}_w^{(1)})^2 = 0.12$ , (iii) Newton-Raphson with  $(\hat{\sigma}_w^{(1)})^2 = 0.14$  and (iv) Newton-Raphson with  $(\hat{\sigma}_w^{(1)})^2 = 0.12$ .

**Example 11.** Consider the problem of calculating the initial alignment of an inertial navigation system. Alignment is the process of estimating the Earth rotation rate and rotating the attitude direction cosine matrix, so that it transforms the body-frame sensor signals to a locally-level frame, wherein certain components of accelerations and velocities approach zero when the platform is

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“Four years ago nobody but nuclear physicists had ever heard of the Internet. Today even my cat, Socks, has his own web page. I’m amazed at that. I meet kids all the time, been talking to my cat on the Internet.” *William Jefferson (Bill) Clinton*

stationary. This can be achieved by a Kalman filter that uses the model (23), where  $x_k \in \mathbb{R}^4$  comprises the errors in earth rotation rate, tilt, velocity and position vectors respectively, and  $w_k \in \mathbb{R}^4$  is a deterministic signal which is a nonlinear function of the states (see [24]). The state matrix is calculated as  $A = I + \Phi T_s + \frac{1}{2!}(\Phi T_s)^2 + \frac{1}{3!}(\Phi T_s)^3$ , where  $T_s$  is the sampling interval,  $\Phi =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ is a continuous-time state matrix and } g \text{ is the universal}$$

gravitational constant. The output mapping within (24) is  $C = [0 \ 0 \ 0 \ 1]$ . Raw three-axis accelerometer and gyro data was recorded from a stationary Litton LN270 Inertial Navigation System at a 500 Hz data rate. In order to generate a compact plot, the initial variance estimates were selected to be 10 times the steady state values.

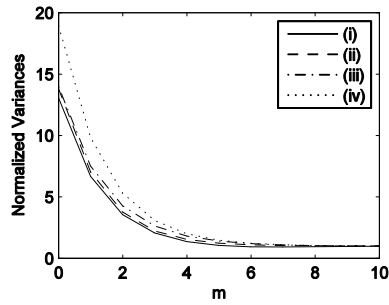


Fig. 3. (i)  $\hat{\sigma}_{1,w}^2$ , (ii)  $\hat{\sigma}_{2,w}^2$ , (iii)  $\hat{\sigma}_{3,w}^2$  and (iv)  $\hat{\sigma}_{4,w}^2$ , normalised by their steady state values, versus EM iteration number for Example 11.

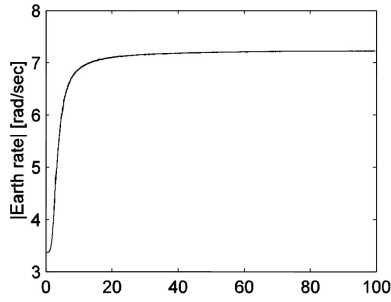


Fig. 4. Estimated magnitude of Earth rotation rate for Example 11

The estimated variances after 10 EM iterations are shown in Fig. 3. The figure demonstrates that approximate MLEs (46) approach steady state values from above, which is consistent with Lemma 5. The estimated Earth rotation rate magnitude versus time is shown in Fig. 4. At 100 seconds, the estimated magnitude of the Earth rate is 72.53 micro-radians per second, that is, one revolution every 24.06 hours. This estimated Earth rate is about 0.5% in error compared with the mean sidereal day of 23.93 hours [25]. Since the estimated Earth rate is in reasonable agreement, it is suggested that the MLEs for the unknown variances are satisfactory (see [19] for further details).

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“On the Internet, nobody knows you’re a dog.” *Peter Steiner*

### 8.3.4 State Matrix Estimation

#### 8.3.4.1 EM Algorithm

The components of the states within (23) are now written as

$$x_{i,k+1} = \sum_{j=1}^n a_{i,j} x_{i,k} + w_{i,k}, \quad (51)$$

where  $a_{i,j}$  denotes the element in row  $i$  and column  $j$  of  $A$ . Consider the problem of estimating  $a_{i,j}$  from samples of  $x_{i,k}$ . The assumption  $x_{i,k+1} \sim \mathcal{N}(\sum_{j=1}^n a_{i,j} x_{i,k}, \sigma_{i,w}^2)$ , leads to the log-likelihood

$$\begin{aligned} & \log f(a_{i,j} | x_{j,k+1}) \\ &= -\frac{N}{2} \log 2\pi - \frac{N}{2} \log \sigma_{i,w}^2 - \frac{1}{2} \sigma_{i,w}^{-2} \sum_{k=1}^N \left( x_{i,k+1} - \sum_{j=1}^n a_{i,j} x_{i,k} \right)^2. \end{aligned} \quad (52)$$

By setting  $\frac{\partial \log f(a_{i,j} | x_{j,k+1})}{\partial a_{i,j}} = 0$ , an MLE for  $a_{i,j}$  is obtained as [20]

$$\hat{a}_{i,j} = \frac{\sum_{k=1}^N \left( x_{i,k+1} - \sum_{j=1, j \neq i}^n a_{i,j} x_{i,k} \right) x_{j,k}}{\sum_{k=1}^N x_{j,k}^2}. \quad (53)$$

Incidentally, the above estimate can also be found using the least-squares method [2], [10] and minimising the cost function  $\sum_{k=1}^N \left( x_{i,k+1} - \sum_{j=1}^n a_{i,j} x_{i,k} \right)^2$ . The expectation of  $\hat{a}_{i,j}$  is [20]

$$E\{\hat{a}_{i,j}\} = E \left\{ \frac{\sum_{k=1}^N \left( \sum_{j=1}^n a_{i,j} x_{i,k} + w_{i,k} - \sum_{j=1, j \neq i}^n a_{i,j} x_{i,k} \right) x_{j,k}}{\sum_{k=1}^N x_{j,k}^2} \right\}$$

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“It’s important for us to explain to our nation that life is important. It’s not only the life of babies, but it’s life of children living in, you know, the dark dungeons of the internet.” *George Walker Bush*



$$= a_{i,j} + E \left\{ \frac{\sum_{k=1}^N w_{i,k} x_{j,k}}{\sum_{k=1}^N x_{j,k}^2} \right\}$$

$$= a_{i,j} ,$$

Since  $w_{i,k}$  and  $x_{i,k}$  are independent. Hence, the MLE (53) is unbiased.

Suppose that an estimate  $\hat{A}^{(u)} = \{a_{i,j}^{(u)}\}$  of  $A$  is available at an iteration  $u$ . The predicted state estimates within (29) can be calculated from

$$\hat{x}_{k+1/k}^{(u)} = (\hat{A}^{(u)} - K_k^{(u)} C) \hat{x}_{k+1/k}^{(u)} + K_k^{(u)} z_k , \quad (54)$$

where  $K_k^{(u)} = \hat{A}^{(u)} P_{k/k-1}^{(u)} C^T (C P_{k/k-1}^{(u)} C^T + R)^{-1}$ , in which  $P_{k/k-1}^{(u)}$  is obtained from the design Riccati equation

$$P_{k+1/k}^{(u)} = (\hat{A}^{(u)} - K_k^{(u)} C) P_{k/k-1}^{(u)} (\hat{A}^{(u)} - K_k^{(u)} C)^T + K_k^{(u)} R (K_k^{(u)})^T + Q . \quad (55)$$

An approximate MLE for  $a_{i,j}$  is obtained by replacing  $x_k$  by  $\hat{x}_{k/k}^{(u)}$  within (53) which results in

$$\hat{a}_{i,j}^{(u+1)} = \frac{\sum_{k=1}^N \left( \hat{x}_{i,k+1/k+1}^{(u)} - \sum_{j=1, j \neq i}^n \hat{a}_{i,j}^{(u)} \hat{x}_{i,k/k}^{(u)} \right) \hat{x}_{j,k/k}^{(u)}}{\sum_{k=1}^N (\hat{x}_{j,k/k}^{(u)})^2} . \quad (56)$$

An iterative procedure for re-estimating an unknown state matrix is proposed below.

**Procedure 3 [20].** Assume that there exists an initial estimate  $\hat{A}^{(1)}$  satisfying  $|\lambda_i(\hat{A}^{(1)})| < 1$ ,  $i = 1, \dots, n$ . Subsequent estimates are calculated using the following two-step EM algorithm.

Step 1. Operate the Kalman filter (29) using (54) on the measurements  $z_k$  over  $k \in [1, N]$  to obtain corrected state estimates  $\hat{x}_{k/k}^{(u)}$  and  $\hat{x}_{k+1/k+1}^{(u)}$ .

Step 2. Copy  $\hat{A}^{(u)}$  to  $\hat{A}^{(u+1)}$ . Use  $\hat{x}_{k/k}^{(u)}$  within (56) to obtain candidate estimates  $\hat{a}_{i,j}^{(u+1)}$ ,  $i, j = 1, \dots, n$ . Include  $\hat{a}_{i,j}^{(u+1)}$  within  $\hat{A}^{(u+1)}$  if  $|\lambda_i(\hat{A}^{(u+1)})| < 1$ ,  $i = 1, \dots, n$ .

The condition  $|\lambda_i(\hat{A}^{(u+1)})| < 1$  within Step 2 ensures that the estimated system is asymptotically stable.

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“The internet is like a gold-rush; the only people making money are those who sell the pans.” *Will Hobbs*

### 8.3.4.2 Properties

The design Riccati difference equation (55) can be written as

$$P_{k+1/k}^{(u)} = (A - K_k^{(u)}C)P_{k/k-1}^{(u)}(A - K_k^{(u)}C)^T + K_k^{(u)}R(K_k^{(u)})^T + Q + S_k^{(u)}, \quad (57)$$

where

$$S_k^{(u)} = (\hat{A}^{(u)} - K_k^{(u)}C)P_{k/k-1}^{(u)}(\hat{A}^{(u)} - K_k^{(u)}C)^T - (A - K_k^{(u)}C)P_{k/k-1}^{(u)}(A - K_k^{(u)}C)^T \quad (58)$$

accounts for the presence of modelling error. In the following, the notation of Lemma 1 is employed to argue that a monotonically non-increasing state matrix estimate sequence results in monotonically non-increasing error covariance sequences.

**Lemma 7 [20].** *In respect of Procedure 3 for estimating  $A$ , suppose the following:*

- (i) *the data  $z_k$  has been generated by (23) – (25) in which  $B$ ,  $C$ ,  $Q$ ,  $R$  are known;*
- (ii)  *$|\lambda_i(\hat{A}^{(1)})| < 1$ ,  $i = 1, \dots, n$ , the pair  $(A, C)$  is observable;*
- (iii) *there exist  $\hat{A}^{(1)} \geq A$  and  $P_{1/0}^{(u+1)} \leq P_{1/0}^{(u)}$  (or  $P_{1/0}^{(u)} \leq P_{1/0}^{(u+1)}$ ) for all  $u > 1$ .*

*Then:*

- (i)  *$\Sigma_{k+1/k}^{(u)} \leq P_{k+1/k}^{(u)}$  (or  $P_{k+1/k}^{(u)} \leq \Sigma_{k+1/k}^{(u)}$ );*
- (ii)  *$\Sigma_{k/k}^{(u)} \leq P_{k/k}^{(u)}$  (or  $P_{k/k}^{(u)} \leq \Sigma_{k/k}^{(u)}$ );*
- (iii)  *$\hat{A}^{(u+1)} \leq \hat{A}^{(u)}$  which implies  $P_{k+1/k}^{(u+1)} \leq P_{k+1/k}^{(u)}$  ( $\hat{A}^{(u)} \leq \hat{A}^{(u+1)}$  which implies  $P_{k+1/k}^{(u)} \leq P_{k+1/k}^{(u+1)}$ )*

*for all  $u \geq 1$ .*

The proof follows *mutatis mutandis* from that of Lemma 1. A heuristic argument is outlined below which suggests that non-increasing error variances lead to a non-increasing state matrix estimate sequence. Suppose that there exists a residual error  $s_k^{(u)} \in \mathbb{R}^n$  at iteration  $u$  such that

$$\hat{x}_{k+1/k+1}^{(u)} = \hat{A}^{(u)}\hat{x}_{k/k}^{(u)} + s_k^{(u)}. \quad (59)$$

The components of (59) are denoted by

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“It may not always be profitable at first for businesses to be online, but it is certainly going to be unprofitable not to be online.” *Ester Dyson*

$$\hat{x}_{i,k+1/k+1}^{(u)} = \sum_{j=1}^n a_{i,j}^{(u)} \hat{x}_{i,k/k}^{(u)} + s_{i,k}^{(u)}, \quad (60)$$

where  $s_{i,k}^{(u)}$  is the  $i^{\text{th}}$  element of  $s_k^{(u)}$ . It follows from (60) and (48) that

$$s_k^{(u)} = L_k^{(u)} (C\tilde{x}_{k+1/k}^{(u)} + v_{k+1}) \quad (61)$$

and

$$s_{i,k}^{(u)} = L_{i,k}^{(u)} (C\tilde{x}_{k+1/k}^{(u)} + v_{k+1}). \quad (62)$$

Using (61) and (63) within (57) yields

$$\begin{aligned} \hat{a}_{i,j}^{(u+1)} &= \hat{a}_{i,j}^{(u)} + \left( \sum_{k=1}^N s_{i,k}^{(u)} \hat{x}_{j,k/k}^{(u)} \right) \left( \sum_{k=1}^N (\hat{x}_{j,k/k}^{(u)})^2 \right)^{-1} \\ &= \hat{a}_{i,j}^{(u)} + L_{i,k}^{(u)} C \left( \sum_{k=1}^N (\tilde{x}_{k+1/k}^{(u)} + C^{\#} v_{k+1}) \hat{x}_{j,k/k}^{(u)} \right) \left( \sum_{k=1}^N (\hat{x}_{j,k/k}^{(u)})^2 \right)^{-1}, \end{aligned} \quad (63)$$

where  $C^{\#}$  denotes the Moore-Penrose pseudo-inverse of  $C$ . It is shown in Lemma 2 under prescribed conditions that  $L^{(u+1)}C \leq L^{(u)}C \leq I$ . Since the non-increasing sequence  $L^{(u)}C$  is a factor of the second term on the right-hand-side of (63), the sequence  $\hat{a}_{i,j}^{(u+1)}$  is expected to be non-increasing.

**Lemma 8 [20]:** Under the conditions of Lemma 7, suppose that  $C$  is full rank, then

$$\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{A}^{(u+1)} = A. \quad (64)$$

**Proof:** It is straight forward to show that  $\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} L_{i,k} C = I$  and therefore  $\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{x}_{k/k}^{(u)} = x_k$ , which implies (64) since the MLE (53) is unbiased.

An illustration is presented below.

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“The Internet is the first thing that humanity has built that humanity doesn't understand, the largest experiment in anarchy that we have ever had.” *Eric Schmidt, Google*

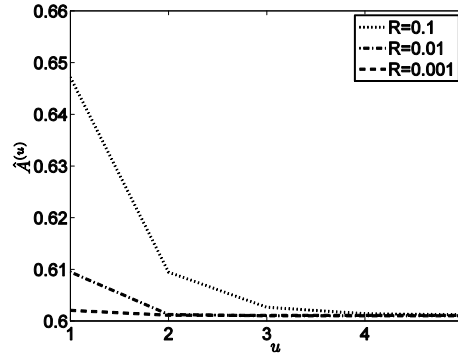


Fig. 5. Sequence of  $\hat{A}^{(u)}$  versus iteration number for Example 12.

**Example 12.** In respect of the model (23) – (25), suppose that  $B = C = 1$ ,  $\sigma_w^2 = 0.2$  are known and  $A = 0.6$  is unknown. Simulations were conducted with 100 realisations of Gaussian process noise and measurement noise of length  $N = 500,000$  for  $R = 0.1, 0.01$  and  $0.001$ . The EM algorithms were initialised with  $\hat{A}^{(1)} = 0.9999$ . It was observed that the resulting estimate sequences were all monotonically decreasing, however, this becomes imperceptible at  $R = 0.001$ , due to the limited resolution of the plot. The mean estimates are shown in Fig. 5. As expected from Lemma 8,  $\hat{A}^{(u)}$  asymptotically approaches the true value of  $A = 0.6$  when the measurement noise becomes negligible.

## 8.4 Smoothing EM Algorithms

### 8.4.1 Process Noise Variance Estimation

#### 8.4.1.1 EM Algorithm

In the previous EM algorithms, the expectation step involved calculating filtered estimates. Similar EM procedures are outlined here where smoothed estimates are used at iteration  $u$  within the expectation step. The likelihood functions described in Sections 8.2.2 and 8.2.3 are exact, provided that the underlying assumptions are correct and actual random variables are available. Under these conditions, the ensuing parameter estimates maximise the likelihood functions and their limit of precision is specified by the associated CRLBs. However, the use of filtered or smoothed quantities leads to approximate likelihood functions, MLEs and CRLBs. It turns out that the approximate MLEs approach the true parameter values under prescribed SNR conditions. It will be shown that the use of smoothed (as opposed

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“The web as I envisaged it, we have not seen as yet. The future is so much bigger than the past.” *Tim Berners-Lee*

to filtered) quantities results in smaller approximate CRLBs, which suggests improved parameter estimation accuracy.

Suppose that the system  $\mathcal{G}$  having the realisation (23) – (24) is non-minimum phase and  $D$  is of full rank. Under these conditions  $\mathcal{G}^{-1}$  exists and the minimum-variance smoother (described in Chapter 7) may be employed to produce input estimates. Assume that an estimate  $\hat{Q}^{(u)} = \text{diag}((\hat{\sigma}_{1,w}^{(u)})^2, (\hat{\sigma}_{2,w}^{(u)})^2, \dots, (\hat{\sigma}_{n,w}^{(u)})^2)$  of  $Q$  is available at iteration  $u$ . The smoothed input estimates,  $\hat{w}_{k/N}^{(u)}$ , are calculated from

$$\begin{bmatrix} x_{k+1/k}^{(u)} \\ \alpha_k^{(u)} \end{bmatrix} = \begin{bmatrix} A_k - K_k^{(u)} C_k & K_k^{(u)} \\ -(\Omega_k^{(u)})^{-1/2} C_k & (\Omega_k^{(u)})^{-1/2} \end{bmatrix} \begin{bmatrix} x_{k/k-1}^{(u)} \\ z_k \end{bmatrix}, \quad (65)$$

$$\begin{bmatrix} \xi_{k-1}^{(u)} \\ \gamma_{k-1}^{(u)} \\ \hat{w}_{k-1/N}^{(u)} \end{bmatrix} = \begin{bmatrix} A_k^T - C_k^T (K_k^{(u)})^T & 0 & C_k^T (\Omega_k^{(u)})^{-1/2} \\ C_k^T (K_k^{(u)})^T & A_k^T & -C_k^T (\Omega_k^{(u)})^{-1/2} \\ -\hat{Q}_k^{(u)} D_k^T (K_k^{(u)})^T & -\hat{Q}_k^{(u)} B_k^T & \hat{Q}_k^{(u)} D_k^T (\Omega_k^{(u)})^{-1/2} \end{bmatrix} \begin{bmatrix} \xi_k^{(u)} \\ \gamma_k^{(u)} \\ \alpha_k^{(u)} \end{bmatrix}, \quad (66)$$

where  $K_k^{(u)} = (A_k P_{k/k-1}^{(u)} C_k^T + B_k \hat{Q}_k^{(u)} D_k^T) (\Omega_k^{(u)})^{-1}$ ,  $\Omega_k^{(u)} = C_k P_{k/k-1}^{(u)} C_k^T + D_k \hat{Q}_k^{(u)} D_k^T + R_k$  and  $P_{k/k-1}^{(u)}$  evolves from the Riccati difference equation  $P_{k+1/k}^{(u)} = A_k P_{k/k-1}^{(u)} A_k^T - (A_k P_{k/k-1}^{(u)} C_k^T + B_k \hat{Q}_k^{(u)} D_k^T) (C_k P_{k/k-1}^{(u)} C_k^T + D_k \hat{Q}_k^{(u)} D_k^T + R_k)^{-1} (C_k P_{k/k-1}^{(u)} A_k^T + D_k \hat{Q}_k^{(u)} B_k^T) + B_k \hat{Q}_k^{(u)} B_k^T$ . A smoothing EM algorithm for iteratively re-estimating  $\hat{Q}^{(u)}$  is described below.

**Procedure 4.** Suppose that an initial estimate  $\hat{Q}^{(1)} = \text{diag}((\hat{\sigma}_{1,w}^{(1)})^2, (\hat{\sigma}_{2,w}^{(1)})^2, \dots, (\hat{\sigma}_{n,w}^{(1)})^2)$  is available. Then subsequent estimates  $\hat{Q}^{(u)}$ ,  $u > 1$ , are calculated by repeating the following two steps.

- Step 1. Use  $\hat{Q}^{(u)} = \text{diag}((\hat{\sigma}_{1,w}^{(u)})^2, (\hat{\sigma}_{2,w}^{(u)})^2, \dots, (\hat{\sigma}_{n,w}^{(u)})^2)$  within (65) – (66) to calculate smoothed input estimates  $\hat{w}_{k/N}^{(u)}$ .
- Step 2. Calculate the elements of  $\hat{Q}^{(u+1)} = \text{diag}((\hat{\sigma}_{1,w}^{(u+1)})^2, (\hat{\sigma}_{2,w}^{(u+1)})^2, \dots, (\hat{\sigma}_{n,w}^{(u+1)})^2)$  using  $\hat{w}_{k/N}^{(u)}$  from Step 1 instead of  $w_k$  within the MLE formula (46).

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“The Internet is not a thing, a place, a single technology, or a mode of governance. It is an agreement.”  
John Gage, Sun Microsystems

### 8.4.1.2 Properties

In the following it is shown that the variance estimates arising from the above procedure result in monotonic error covariances. The additional term within the design Riccati difference equation (57) that accounts for the presence of parameter error is now given by  $S_k^{(u)} = B(\hat{Q}^{(u)} - Q)B^T$ . Let  $\hat{\Delta}^{(u)}$  denote an approximate spectral factor arising in the design of a smoother using  $P_{k/k-1}^{(u)}$  and  $K_k^{(u)}$ . Employing the notation and approach of Chapter 7, it is straightforward to show that

$$\hat{\Delta}^{(u)}(\hat{\Delta}^{(u)})^H = \Delta\Delta^H + C_k \mathcal{G}_0 (P_{k/k-1}^{(u)} - P_{k+1/k}^{(u)} + S^{(u)}) \mathcal{G}_0^H C_k^T. \quad (67)$$

Define the stacked vectors  $v = [v_1^T, \dots, v_K^T]^T$ ,  $w = [w_1^T, \dots, w_N^T]^T$ ,  $\hat{w}^{(u)} = [(\hat{w}_{1/N}^{(u)})^T, \dots, (\hat{w}_{N/N}^{(u)})^T]^T$  and  $\tilde{w}^{(u)} = w - \hat{w}^{(u)} = [(\tilde{w}_{1/N}^{(u)})^T, \dots, (\tilde{w}_{N/N}^{(u)})^T]^T$ . The input estimation error is generated by  $\tilde{w}^{(u)} = \mathcal{R}^{(u)} \begin{bmatrix} v \\ w \end{bmatrix}$ , where  $\mathcal{R}_{ei}^{(u)}(\mathcal{R}_{ei}^{(u)})^H = \mathcal{R}_{ei1}^{(u)}(\mathcal{R}_{ei1}^{(u)})^H + \mathcal{R}_{ei2}^{(u)}(\mathcal{R}_{ei2}^{(u)})^H$ , in which

$$\mathcal{R}_{ei2}^{(u)} = Q\mathcal{G}^H \left( (\hat{\Delta}^{(u)}(\hat{\Delta}^{(u)})^H)^{-1} - (\Delta\Delta^H)^{-1} \right) \Delta, \quad (68)$$

and  $\mathcal{R}_{ei1}^{(u)}(\mathcal{R}_{ei1}^{(u)})^H = Q\mathcal{G}_2^H - Q\mathcal{G}^H(\Delta\Delta^H)^{-1}\mathcal{G}Q$ . It is shown in the lemma below that the sequence  $\|\tilde{w}^{(u)}(\tilde{w}^{(u)})^T\|_2 = \|\mathcal{R}_{ei}^{(u)}(\mathcal{R}_{ei}^{(u)})^H\|_2$  is monotonically non-increasing or monotonically non-decreasing, depending on the initial conditions.

**Lemma 9:** *In respect of Procedure 4 for estimating  $Q$ , suppose the following:*

- (i) *the system (23) – (24) is non-minimum phase, in which  $A, B, C, D, R$  are known,  $|\lambda_i(\hat{A}^{(1)})| < 1$ ,  $i = 1, \dots, n$ , the pair  $(A, C)$  is observable and  $D$  is of full rank;*
- (ii) *the solutions  $P_{1/0}^{(1)}, P_{1/0}^{(2)}$  of (57) for  $\hat{Q}^{(2)} \geq \hat{Q}^{(1)}$  satisfy  $P_{1/0}^{(2)} \leq P_{1/0}^{(1)}$  (or the solutions  $P_{1/0}^{(1)}, P_{1/0}^{(2)}$  of (57) for  $\hat{Q}^{(1)} \geq \hat{Q}^{(2)}$  satisfy  $P_{1/0}^{(1)} \leq P_{1/0}^{(2)}$ ).*

*Then:*

- (i)  $P_{k+1/k}^{(u)} \leq P_{k/k-1}^{(u)}$  (or  $P_{k/k-1}^{(u)} \leq P_{k+1/k}^{(u)}$ ) for all  $k, u \geq 1$ ;
- (ii)  $P_{k+1/k}^{(u+1)} \leq P_{k+1/k}^{(u)}$  and  $P_{k/k-1}^{(u+1)} \leq P_{k/k-1}^{(u)}$  (or  $P_{k+1/k}^{(u)} \leq P_{k+1/k}^{(u+1)}$  and  $P_{k/k-1}^{(u)} \leq P_{k/k-1}^{(u+1)}$ ) for all  $k, u \geq 1$ ;

---

“The spread of computers and the internet will put jobs in two categories. People who tell computers what to do, and people who are told by computers what to do.” *Marc Andreessen*

$$(iii) \left\| \mathcal{R}_{ei}^{(u+1)} (\mathcal{R}_{ei}^{(u+1)})^H \right\|_2 \leq \left\| \mathcal{R}_{ei}^{(u)} (\mathcal{R}_{ei}^{(u)})^H \right\|_2 \quad (or \quad \left\| \mathcal{R}_{ei}^{(u)} (\mathcal{R}_{ei}^{(u)})^H \right\|_2 \leq \left\| \mathcal{R}_{ei}^{(u+1)} (\mathcal{R}_{ei}^{(u+1)})^H \right\|_2) \text{ for } u \geq 1.$$

**Proof:** (i) and (ii) This follows from  $S^{(u+1)} \leq S^{(u)}$  within condition (iii) of Theorem 2 of Chapter 8. Since  $\mathcal{R}_{ei1}^{(u)} (\mathcal{R}_{ei1}^{(u)})^H$  is common to  $\mathcal{R}_{ei}^{(u)} (\mathcal{R}_{ei}^{(u)})^H$  and  $\mathcal{R}_{ei}^{(u+1)} (\mathcal{R}_{ei}^{(u+1)})^H$ , it suffices to show that

$$\left\| \mathcal{R}_{ei2}^{(u+1)} (\mathcal{R}_{ei2}^{(u+1)})^H \right\|_2 \leq \left\| \mathcal{R}_{ei2}^{(u)} (\mathcal{R}_{ei2}^{(u)})^H \right\|_2. \quad (69)$$

Substituting (67) into (68) yields

$$\mathcal{R}_{ei2}^{(u)} = Q \mathcal{G}^H \left( \Delta \Delta^H + C_k \mathcal{G}_0 (P_{k/k-1}^{(u)} - P_{k+1/k}^{(u)} + S^{(u)}) \mathcal{G}_0^H C_k^T \right)^{-1} - (\Delta \Delta^H)^{-1} \Delta. \quad (70)$$

Note for linear time-invariant systems  $X, Y_1 \geq Y_2$ , that

$$(XX^H)^{-1} - (XX^H + Y_1)^{-1} \geq (XX^H)^{-1} - (XX^H + Y_2)^{-1}. \quad (71)$$

Since  $\left\| \mathcal{G}_0 (P_{k/k-1}^{(u+1)} - P_{k+1/k}^{(u+1)} + S^{(u+1)}) \mathcal{G}_0^H \right\|_2 \leq \left\| \mathcal{G}_0 (P_{k/k-1}^{(u)} - P_{k+1/k}^{(u)} + S^{(u)}) \mathcal{G}_0^H \right\|_2$ , (69) follows from (70) and (71).

As is the case for the filtering EM algorithm, the process noise variance estimates asymptotically approach the exact values when the SNR is sufficiently high.

**Lemma 10:** Under the conditions of Lemma 9,

$$\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{Q}^{(u)} = Q. \quad (72)$$

**Proof:** By inspection of the input estimator,  $\mathcal{H}_{IE} = Q \mathcal{G}^H (\Delta \Delta^H)^{-1} = Q \mathcal{G}^H (\mathcal{G} Q \mathcal{G}^H + R)^{-1}$ , it follows that  $\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \mathcal{H}_{IE} = \mathcal{G}^{-1}$  and therefore

$\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{w}_{k/N}^{(u)} = w_k$ , which implies (72), since the MLE (46) is unbiased for large  $N$ .

It is observed anecdotally that the variance estimates produced by the above smoothing EM algorithm are more accurate than those from the corresponding filtering procedure. This is consistent with the following comparison of approximate CRLBs.

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“What the country needs are a few labour-making inventions.” *Arnold H. Glasow*

**Lemma 11:**

$$-\left(\frac{\partial^2 \log f(\sigma_{i,w}^2 | \hat{x}_{k/N})}{(\partial \sigma_{i,w}^2)^2}\right)^{-1} < -\left(\frac{\partial^2 \log f(\sigma_{i,w}^2 | \hat{x}_{k/k})}{(\partial \sigma_{i,w}^2)^2}\right)^{-1}. \quad (73)$$

**Proof:** The vector state elements within (23) can be written in terms of smoothed state estimates,  $x_{i,k+1} = A_i \hat{x}_{k/N} + w_{i,k} = A_i x_k + w_{i,k} - A_i \tilde{x}_{k/N}$ , where  $\tilde{x}_{k/N} = x_k - \hat{x}_{k/N}$ . From the approach of Example 8, the second partial derivative of the corresponding approximate log-likelihood function with respect to the process noise variance is

$$\frac{\partial^2 \log f(\sigma_{i,w}^2 | \hat{x}_{k/K})}{(\partial \sigma_{i,w}^2)^2} = -\frac{N}{2} (\sigma_{i,w}^2 + A_i E\{\tilde{x}_{k/N} \tilde{x}_{k/N}^T\} A_i^T)^{-2}.$$

Similarly, the use of filtered state estimates leads to

$$\frac{\partial^2 \log f(\sigma_{i,w}^2 | \hat{x}_{k/k})}{(\partial \sigma_{i,w}^2)^2} = -\frac{N}{2} (\sigma_{i,w}^2 + A_i E\{\tilde{x}_{k/k} \tilde{x}_{k/k}^T\} A_i^T)^{-2}.$$

The minimum-variance smoother minimises both the causal part and the non-causal part of the estimation error, whereas the Kalman filter only minimises the causal part. Therefore,  $E\{\tilde{x}_{k/N} \tilde{x}_{k/N}^T\} < E\{\tilde{x}_{k/k} \tilde{x}_{k/k}^T\}$ . Thus, the claim (73) follows.

## 8.4.2 State Matrix Estimation

### 8.4.2.2 EM Algorithm

Smoothed state estimates are obtained from the smoothed inputs via

$$\hat{x}_{k+1/N}^{(u)} = A_k \hat{x}_{k/N}^{(u)} + B_k \hat{w}_{k/N}^{(u)}. \quad (74)$$

The resulting  $\hat{x}_{k/N}^{(u)}$  are used below to iteratively re-estimate state matrix elements.

**Procedure 5.** Assume that there exists an initial estimate  $\hat{A}^{(1)}$  of  $A$  such that  $|\lambda_i(\hat{A}^{(1)})| < 1$ ,  $i = 1, \dots, n$ . Subsequent estimates,  $\hat{A}^{(u)}$ ,  $u > 1$ , are calculated using the following two-step EM algorithm.

Step 1. Operate the minimum-variance smoother recursions (65), (66), (74) designed with  $\hat{A}^{(u)}$  to obtain  $\hat{x}_{k/N}^{(u)}$ .

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“Big data will replace the need for 80% of all doctors.” Vinod Khosla



Step 2. Copy  $\hat{A}^{(u)}$  to  $\hat{A}^{(u+1)}$ . Use  $\hat{x}_{k/N}^{(u)}$  instead of  $x_k$  within (53) to obtain candidate estimates  $\hat{a}_{i,j}^{(u+1)}$ ,  $i, j = 1, \dots, n$ . Include  $\hat{a}_{i,j}^{(u+1)}$  within  $\hat{A}^{(u+1)}$  if  $|\lambda_i(\hat{A}^{(u+1)})| < 1$ ,  $i = 1, \dots, n$ .

### 8.4.2.3 Properties

Denote  $x = [x_1^T, \dots, x_N^T]^T$ ,  $\hat{x}^{(u)} = [(\hat{x}_{1/N}^{(u)})^T, \dots, (\hat{x}_{N/N}^{(u)})^T]^T$  and  $\tilde{x}^{(u)} = x - \hat{x}^{(u)} = [(\tilde{x}_{1/N}^{(u)})^T, \dots, (\tilde{x}_{N/N}^{(u)})^T]^T$ . Let  $\mathcal{R}^{(u)}$  be redefined as the system that maps the inputs  $\begin{bmatrix} v \\ w \end{bmatrix}$  to smoother state estimation error  $\tilde{x}^{(u)}$ , that is,  $\tilde{x}^{(u)} = \mathcal{R}^{(u)} \begin{bmatrix} v \\ w \end{bmatrix}$ . It is stated below that the estimated state matrix iterates result in a monotonic sequence of state error covariances.

**Lemma 12:** *In respect of Procedure 5 for estimating  $A$  and  $x$ , suppose the following:*

- (i) *the system (23) – (24) is non-minimum phase, in which  $B, C, D, Q, R$  are known,  $|\lambda_i(\hat{A}^{(u+1)})| < 1$ , the pair  $(A, C)$  is observable and  $D$  is of full rank;*
- (ii) *there exist solutions  $P_{1/0}^{(1)}, P_{1/0}^{(2)}$  of (57) for  $AA^T \leq \leq$  satisfying  $\leq$  (or the solutions, of (31) for  $\leq \leq AA^T$  satisfying  $\leq$ ).*

*Then  $\leq$  (or  $\leq$ ) for  $u \geq 1$ .*

The proof is omitted since it follows *mutatis mutandis* from that of Lemma 9. Suppose that the smoother (65), (66) designed with the estimates is employed to calculate input estimates. An approximate log-likelihood function for the unknown given samples of is

$$\log f(a_{i,j} | \hat{w}_{i,k/K}^{(u)}) = -\frac{N}{2} \log 2\pi - \frac{N}{2} \log(\sigma_{i,w}^{(u)})^2 - \frac{1}{2} (\sigma_{i,w}^{(u)})^{-2} \sum_{k=1}^N \hat{w}_{i,k/K}^{(u)} (\hat{w}_{i,k/N}^{(u)})^T. \quad (75)$$

Now let  $\mathcal{R}^{(u)}$  denote the map from  $\begin{bmatrix} v \\ w \end{bmatrix}$  to the smoother input estimation error  $\tilde{w}^{(u)} = w - \hat{w}^{(u)}$  at iteration  $u$ . It is argued below that the sequence of state matrix iterates maximises (75).

**Lemma 13:** *Under the conditions of Lemma 12,  $\|\mathcal{R}^{(u+1)}(\mathcal{R}^{(u+1)})^H\|_2 \leq \|\mathcal{R}^{(u)}(\mathcal{R}^{(u)})^H\|_2$  for  $u \geq 1$ .*

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“Things like chatbots, machine learning tools, natural language processing, or sentiment analysis are applications of artificial intelligence that may one day profoundly change how we think about and transact in travel and local experiences.” *Gillian Tans, CEO of Booking.com*

The proof follows *mutatis mutandis* from that of Lemma 9. The above Lemma implies

$$E\{\tilde{w}^{(u+1)}(\tilde{w}^{(u+1)})^T\} \leq E\{\tilde{w}^{(u)}(\tilde{w}^{(u)})^T\}. \quad (76)$$

It follows from  $\hat{w}^{(u)} = w - \tilde{w}^{(u)}$  that  $E\{\tilde{w}^{(u)}(\tilde{w}^{(u)})^T\} = E\{w + \tilde{w}^{(u)}(w + (\tilde{w}^{(u)})^T) = E\{\tilde{w}^{(u)}(\tilde{w}^{(u)})^T\} + Q$ , which together with (76) implies  $E\{\hat{w}^{(u+1)}(\hat{w}^{(u+1)})^T\} \leq E\{\hat{w}^{(u)}(\hat{w}^{(u)})^T\}$  and  $\log f(a_{i,jw} | \hat{w}_{i,k/N}^{(u+1)}) \geq \log f(a_{i,jw} | \hat{w}_{i,k/N}^{(u)})$  for all  $u \geq 1$ . Therefore, it is expected that the sequence of state matrix estimates will similarly vary monotonically. Next, it is stated that the state matrix estimates asymptotically approach the exact values when the SNR is sufficiently high.

**Lemma 14:** *Under the conditions of Lemma 9,*

$$\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{A}^{(u)} = A. \quad (77)$$

**Proof:** *From the proof of Lemma 10,*  $\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0, u \rightarrow \infty} \hat{w}_{k/N}^{(u)} = w_k$ , *therefore, the states within (74) are reconstructed exactly. Thus, the claim (77) follows since the MLE (53) is unbiased.*

It is expected that the above EM smoothing algorithm offers improved state matrix estimation accuracy.

**Lemma 15:**

$$-\left(\frac{\partial^2 \log f(a_{i,j} | \hat{x}_{k/N})}{(\partial a_{i,j})^2}\right)^{-1} \leq -\left(\frac{\partial^2 \log f(a_{i,j} | \hat{x}_{k/k})}{(\partial a_{i,j})^2}\right)^{-1}. \quad (78)$$

**Proof:** *Using smoothed states within (51) yields  $x_{i,k+1} = \sum_{j=1}^n a_{i,j} \hat{x}_{i,k/N} + w_{i,k} = \sum_{j=1}^n a_{i,j} x_{i,k} + w_{i,k} - \sum_{j=1}^n a_{i,j} \tilde{x}_{i,k/N}$ , where  $\tilde{x}_{k/N} = x_k - \hat{x}_{k/N}$ . The second partial derivative of the corresponding log-likelihood function with respect to  $a_{i,j}$  is*

$$\frac{\partial^2 \log f(a_{i,j} | \hat{x}_{k/N})}{(\partial a_{i,j})^2} = -\frac{N}{2} (\sigma_{i,w}^2 + A_i E\{\tilde{x}_{k/N} \tilde{x}_{k/N}^T\} A_i^T)^{-1} \sum_{k=1}^N x_{j,k}^2.$$

*Similarly, the use of filtered state estimates leads to*

---

“The best way to prepare is to write programs, and to study great programs that other people have written. In my case, I went to the garbage cans at the Computer Science Center and I fished out listings of their operating system.” *William Henry (Bill) Gates III*

$$\frac{\partial^2 \log f(a_{i,j} | \hat{x}_{k/k})}{(\partial a_{i,j})^2} = -\frac{N}{2} (\sigma_{i,w}^2 + A_i E\{\tilde{x}_{k/k} \tilde{x}_{k/k}^T\} A_i^T)^{-1} \sum_{k=1}^N x_{j,k}^2.$$

The result (78) follows since  $E\{\tilde{x}_{k/N} \tilde{x}_{k/N}^T\} < E\{\tilde{x}_{k/k} \tilde{x}_{k/k}^T\}$ .

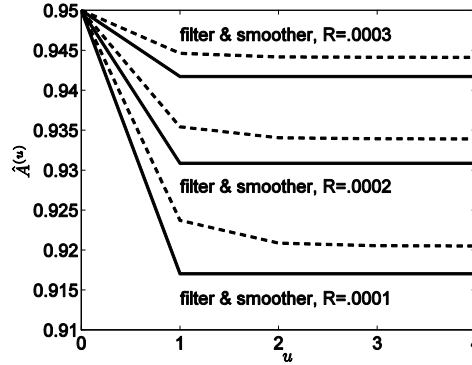


Fig. 6. State matrix estimates calculated by the smoother EM algorithm and filter EM algorithm for Example 13. It can be seen that the  $\hat{A}^{(u)}$  better approach the nominal  $A$  at higher SNR.

**Example 13.:** Consider a system where  $B = C = D = Q = 1$ ,  $R = \{0.0001, 0.0002, 0.0003\}$  are known and  $A = 0.9$  but is unknown. Simulations were conducted using 30 noise realisations with  $N = 500,000$ . The results of the above smoothing EM algorithm and the filtering EM algorithms, initialised with  $\hat{A}^{(0)} = 1.03A$ , are respectively shown by the dotted and dashed lines within Fig. 6. The figure shows that the estimates improve with increasing  $u$ , which is consistent with Lemma 15. The estimates also improve with increasing SNR which illustrates Lemmas 8 and 14. It is observed anecdotally that the smoother EM algorithm outperforms the filter EM algorithm for estimation of  $A$  at high signal-to-noise-ratios.

### 8.4.3 Measurement Noise Variance Estimation

The discussion of an EM procedure for measurement noise variance estimation is presented in a summary form because it follows analogously to the algorithms described previously.

**Procedure 6.** Assume that an initial estimate  $\hat{R}^{(1)}$  of  $R$  is available. Subsequent estimates  $\hat{R}^{(u)}$ ,  $u > 1$ , are calculated by repeating the following two-step procedure.

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“Don't worry about people stealing your ideas. If your ideas are any good, you'll have to ram them down people's throats.” *Howard Hathaway Aiken*

- Step 1. Operate the minimum-variance smoother (7.66), (7.68), (7.69) designed with  $\hat{R}^{(u)}$  to obtain corrected output estimates  $\hat{y}_{k/N}^{(u)}$ .
- Step 2. For  $i = 1, \dots, p$ , use  $\hat{y}_{k/N}^{(u)}$  instead of  $y_k$  within (27) to obtain  $\hat{R}^{(u+1)} = \text{diag}((\hat{\sigma}_{1,v}^{(u+1)})^2, (\hat{\sigma}_{2,v}^{(u+1)})^2, \dots, (\hat{\sigma}_{n,v}^{(u+1)})^2)$ .

It can be shown using the approach of Lemma 9 that the sequence of measurement noise variance estimates are either monotonically non-increasing or non-decreasing depending on the initial conditions. When the SNR is sufficiently low, the measurement noise variance estimates converge to the actual value.

**Lemma 16:** *In respect of Procedure 6,*

$$\lim_{R^{-1} \rightarrow 0, Q \rightarrow 0, u \rightarrow \infty} R^{(u)} = R. \quad (79)$$

**Proof:** By inspection of the output,  $\mathcal{H}_{OE} = \mathcal{G}Q\mathcal{G}^H(\mathcal{G}Q\mathcal{G}^H + R)^{-1}$ , it follows that  $\lim_{R^{-1} \rightarrow 0, Q \rightarrow 0, u \rightarrow \infty} \mathcal{H}_{OE} = 0$ , which together with the observation  $\lim_{R^{-1} \rightarrow 0, Q \rightarrow 0, u \rightarrow \infty} E\{zz^T\} = R$  implies (79), since the MLE (27) is unbiased for large  $N$ .

Once again, the variance estimates produced by the above procedure are expected to be more accurate than those relying on filtered estimates.

**Lemma 17:**

$$-\left(\frac{\partial^2 \log f(\sigma_{i,v}^2 | \hat{y}_{i,k/N})}{(\partial \sigma_{i,v}^2)^2}\right)^{-1} < -\left(\frac{\partial^2 \log f(\sigma_{i,v}^2 | \hat{y}_{i,k/k})}{(\partial \sigma_{i,v}^2)^2}\right)^{-1}. \quad (80)$$

**Proof:** The second partial derivative of the corresponding log-likelihood function with respect to the process noise variance is

$$\frac{\partial^2 \log f(\sigma_{i,v}^2 | \hat{y}_{i,k/N})}{(\partial \sigma_{i,v}^2)^2} = -\frac{N}{2}(\sigma_{i,v}^2 + E\{\tilde{y}_{i,k/N} \tilde{y}_{i,k/N}^T\})^{-2},$$

where  $\tilde{y}_{k/N}^{(u)} = y - \hat{y}_{k/N}^{(u)}$ . Similarly, the use of filtered state estimates leads to

$$\frac{\partial^2 \log f(\sigma_{i,v}^2 | \hat{y}_{i,k/k})}{(\partial \sigma_{i,v}^2)^2} = -\frac{N}{2}(\sigma_{i,v}^2 + E\{\tilde{y}_{i,k/k} \tilde{y}_{i,k/k}^T\})^{-2},$$

where  $\tilde{y}_{k/k}^{(u)} = y - \hat{y}_{k/k}^{(u)}$ . The claim (80) follows since  $E\{\tilde{y}_{k/N} \tilde{y}_{k/N}^T\} < E\{\tilde{y}_{k/k} \tilde{y}_{k/k}^T\}$ .

## 8.5 Unbiased Estimation of State Space Parameters

### 8.5.1 Overview

Although the previously-discussed EM algorithms can yield improved state matrix and input covariance estimates, it has been found that they are only accurate when the measurement noise is negligible – see Lemma 10 and 14. Similarly in subspace identification, least-squares estimation of unknown state-space matrices can lead to biased results when the states are corrupted by noise (see [29] – [31] and the references therein). Techniques have been previously reported for the elimination of bias within autoregressive moving average (ARMA) models, including noise estimation [29] – [31], compensation [32], prefiltering [33] and iterative methods [31].

This section attends to unbiased estimation of state-space parameters. In common with ARMA model parameter estimation [29] – [31], a correction term is introduced to eliminate bias error. Unbiased, consistent, closed-form estimates of a state matrix, an input covariance and a measurement noise covariance are developed. It is established under simplifying conditions that they are equal to MLEs and attain the corresponding CRLBs. A search procedure is described for problems in which both the state matrix and the measurement noise covariance are unknown. The remainder of this section is organised as follows. Section 8.5.2 defines the problem of interest. The estimation of state-space parameters from noisy observations is described in Section 8.5.3. An example that demonstrates filtering performance benefits is presented in Section 8.5.4.

### 8.5.2 Problem Definition

As before, let  $w_k \in \mathbb{R}^n$  denote an uncorrelated white sequence. Consider again a time-invariant state-space system having a realisation of the form

$$x_{k+1} = Ax_k + w_k \quad (81)$$

$$y_k = Cx_k, \quad (82)$$

where  $x_k \in \mathbb{R}^n$  is the internal state,  $y_k \in \mathbb{R}^m$  is the output,  $A, \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{m \times n}$ . Observations of (82) are modelled as

$$z_k = y_k + v_k, \quad (83)$$

where  $v_k \in \mathbb{R}^m$  is an independent, white, measurement noise sequence.

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“The power of an idea can be measured by the degree of resistance it attracts.” *David Yoho*

It is convenient to write (81) - (83) in block form over integer  $k \in [1, N]$  as

$$X_{k+1} = AX_k + W_k \quad (84)$$

$$Y_k = CX_k \quad (85)$$

$$Z_k = Y_k + V_k, \quad (86)$$

$$\text{where } X_{k+1} = \begin{bmatrix} x_{1,2} & \cdots & x_{1,N+1} \\ \vdots & \ddots & \vdots \\ x_{n,2} & \cdots & x_{n,N+1} \end{bmatrix}, X_k = \begin{bmatrix} x_{1,1} & \cdots & x_{1,N} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,N} \end{bmatrix}, W_k = \begin{bmatrix} w_{1,1} & \cdots & w_{1,N} \\ \vdots & \ddots & \vdots \\ w_{n,1} & \cdots & w_{n,N} \end{bmatrix}$$

$$\in \mathbb{R}^{n \times N}, Y_k = \begin{bmatrix} y_{1,1} & \cdots & y_{1,N} \\ \vdots & \ddots & \vdots \\ y_{m,1} & \cdots & y_{m,N} \end{bmatrix}, V_k = \begin{bmatrix} v_{1,1} & \cdots & v_{1,N} \\ \vdots & \ddots & \vdots \\ v_{m,1} & \cdots & v_{m,N} \end{bmatrix} \quad \text{and} \quad Z_k = \begin{bmatrix} z_{1,1} & \cdots & z_{1,N} \\ \vdots & \ddots & \vdots \\ z_{m,1} & \cdots & z_{m,N} \end{bmatrix} \in \mathbb{R}^{m \times N}.$$

Recall that  $(\cdot)^T$  and  $\delta_{j,k} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$  denote the transpose and the Kronecker delta function, respectively. It is convenient to make some assumptions below in which  $\lambda_i(A)$  denotes the  $i^{\text{th}}$  eigenvalue of  $A$  and  $E\{\cdot\}$  is the expectation operator.

**Assumption 1:** (a)  $|\lambda_i(A)| < 1, i = 1, \dots, n$ . (b) The system (1) – (2) is reachable. (c) The pair  $(A, C)$  is observable. (d)  $E\{w_k\} = E\{v_k\} = 0$ . (e)  $E\{w_i w_k^T\} = Q \delta_{j,k}$ . (f)  $E\{v_i v_k^T\} = R \delta_{j,k}$ . (g)  $E\{w_i v_k^T\} = 0$ .

Assumption 1(a) ensures that the system (1) – (2) is asymptotically stable. Assuming reachability, 1(b), means that all modes of the system will be excited. Observability is assumed, 1(c), so that the states can be recovered from the system's output. The zero-mean 1(d) and uncorrelated noise assumptions 1(e) – (g) allow the standard form of the Kalman filter to be employed for calculating minimum mean-square error (MSE) state estimates from measurements (3). Note from (1) and Assumption 1(e) that  $E\{x_{k+1} w_{k+1}^T\} = 0$ , i.e.,  $x_k$  is uncorrelated with  $w_k$ .

This section addresses the problem of estimating unbiased  $A$ ,  $Q$  and  $R$  parameters from noisy measurement sequences. This is motivated by a desire to construct

optimal predictors, filters and smoothers that return better MSE performance than those designed with conventional least-squares parameter estimates.

### 8.5.3 Identification of Model Parameters

#### 8.5.3.1 Prerequisites

The optimal filter gain minimises the predicted and filtered error covariances, which relies on having precise knowledge of parameters  $A$ ,  $C$ ,  $Q$  and  $R$ . If these parameters are not known accurately and a different gain is calculated then the observed error covariances will be larger. That is, estimation performance degrades when the identified model parameters are inaccurate.

Use will be made of the Moore-Penrose pseudo inverse of  $C$ , which is denoted by  $C^\dagger = (C^T C)^{-1} C^T \in \mathbb{R}^{n \times m}$ . This is a generalisation of the matrix inverse and can be calculated from the Singular Value Decomposition. It is commonly used to find a least-squares solution to an overdetermined system of linear equations [34]. An additional assumption is stated which enables  $x_k$  to be estimated directly from  $z_k$  using the pseudo inverse.

**Assumption 2:**  $C$  is full column rank so that  $C^\dagger$  exists and  $C^\dagger C = I$ . Under the assumed condition, state estimates can be calculated as

$$\hat{x}_k = C^\dagger z_k = x_k + C^\dagger v_k, \quad (87)$$

From (87) and Assumption 1(d), note that  $E\{x - \hat{x}\} = 0$ , i.e., the  $\hat{x}_k$  are unbiased.

By defining  $\hat{X}_k = \begin{bmatrix} \hat{x}_{1,1} & \cdots & \hat{x}_{1,N} \\ \vdots & \ddots & \vdots \\ \hat{x}_{n,1} & \cdots & \hat{x}_{n,N} \end{bmatrix}$ ,  $\hat{X}_{k+1} = \begin{bmatrix} \hat{x}_{1,2} & \cdots & \hat{x}_{1,N+1} \\ \vdots & \ddots & \vdots \\ \hat{x}_{n,2} & \cdots & \hat{x}_{n,N+1} \end{bmatrix} \in \mathbb{R}^{n \times N}$ , (87)

may be written in block form

$$\hat{X}_k = C^\dagger Z_k = X_k + C^\dagger V_k. \quad (88)$$

In many applications  $R$  can be estimated from the observations during periods when  $y_k$  is available. Often the  $C$  is known or selected by the filter designers, whereas  $A$  and  $Q$  typically require to be identified. Note that existing subspace identification techniques [12] – [14], [28] may be used to identify an unknown  $C$ . It is shown below that if  $R$  is known, the  $A$  and  $Q$  can be identified from the state

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“You do not really understand something unless you can explain it to your grandmother.” *Albert Einstein*

estimates (87) – (88). Conversely, if  $A$  is known then  $R$  can be identified. A search procedure is advocated for cases where both  $A$  and  $R$  are unknown.

### 8.5.3.2 Identification of $A$ with $R$ Known

An unbiased estimate  $\hat{A} = \{\hat{a}_{i,j}\}$  of  $A = \{a_{i,j}\}$  can be found by either solving or searching for the minima of cost functions, depending on whether  $R$  is known.

Let

$$\hat{X}_{i,k+1} = [\hat{x}_{i,2} \quad \cdots \quad \hat{x}_{i,N+1}] \in \mathbb{R}^{1 \times N}$$

and

$$\hat{A}_i = [\hat{a}_{i,1} \quad \cdots \quad \hat{a}_{i,n}] \in \mathbb{R}^{1 \times n}$$

denote the  $i^{\text{th}}$  rows of  $\hat{X}_{k+1}$  and  $\hat{A}$ , respectively.

**Lemma 18:** Suppose  $R$  is known. The estimate

$$\hat{A} = E\{\hat{x}_{k+1}\hat{x}_k^T\}(E\{\hat{x}_k\hat{x}_k^T\} - C^\dagger RC^{\dagger T})^{-1} \quad (89)$$

minimises the cost functions

$$J_i(\hat{A}_i, C, R, X_k, \hat{X}_{i,k+1}) = \|\hat{X}_{i,k+1} - \hat{A}_i X_k\|_2^2, \quad (90)$$

$i = 1, \dots, n$ , where  $\|\cdot\|_2$  denotes the 2-norm.

**Proof:** Equations (86) and (88) imply  $\hat{X}_k \hat{X}_k^T = X_k X_k^T + X_k V_k^T C^{\dagger T} + C^\dagger V_k X_k^T + C^\dagger RC^{\dagger T}$ , which together with Assumptions 1(f) – (g) give

$$\hat{X}_k \hat{X}_k^T = X_k X_k^T + C^\dagger RC^{\dagger T}. \quad (91)$$

Let  $\nabla_{\hat{A}_i}$  denote the gradient with respect to  $\hat{A}_i$ . Setting  $\nabla_{\hat{A}_i} J_i(\hat{A}_i, C, R, X_k, \hat{X}_{i,k+1}) = -2(\hat{X}_{i,k+1} - \hat{A}_i X_k) X_k^T = 0$ ,  $i = 1, \dots, n$ , leads to

$$\hat{A}_i = \hat{X}_{i,k+1} \hat{X}_k^T (X_k X_k^T)^{-1}. \quad (92)$$

Using (91) within (92) results in

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“From the time I was seven, when I purchased my first calculator, I was fascinated by the idea of a machine that could compute things.” *Michael Dell*



$$\hat{A}_i = E\{\hat{X}_{i,k+1}\hat{X}_{i,k}^T\}(E\{\hat{X}_k\hat{X}_k^T\} - C^\dagger RC^{\dagger T})^{-1}. \quad (93)$$

Stacking the rows (93) of  $\hat{A}$  yields (89).

Thus,  $\hat{A}$  is calculated from (89) in which  $\hat{x}_k$  and  $\hat{x}_{k+1}$  are found using the pseudo inverse in (87). A method for estimating an unknown  $R$  within (89) is described in Section III E.

**Lemma 19:** Under Assumptions 1 - 2, the estimate (89) is unbiased.

**Proof:** (a) Equations (81), (87) yield  $E\{\hat{x}_k\hat{x}_k^T\} = E\{x_kx_k^T\} + E\{x_kv_k^T\}C^{\dagger T} + C^\dagger E\{v_kx_k^T\} + C^\dagger RC^{\dagger T}$ , which together with Assumptions 1(f) – (g) give

$$E\{\hat{x}_k\hat{x}_k^T\} = E\{x_kx_k^T\} + C^\dagger RC^{\dagger T}. \quad (94)$$

Right-multiplying (1) by  $x_k$  and taking expectations results in  $E\{x_{k+1}x_k^T\} = AE\{x_kx_k^T\} + E\{x_kw_k^T\}B^T$ , from which the actual state matrix is specified by

$$A = E\{x_{k+1}x_k^T\}E\{x_kx_k^T\}^{-1}. \quad (95)$$

since  $E\{x_kw_k^T\} = 0$ . It follows from (81), (87) that  $E\{\hat{x}_{k+1}\hat{x}_k^T\} = AE\{x_kx_k^T\} + AE\{x_kv_k^T\}C^{\dagger T} + E\{w_kx_k^T\} + E\{w_kv_k^T\}C^{\dagger T} + C^\dagger E\{v_{k+1}v_k^T\}C^{\dagger T} + C^\dagger E\{v_{k+1}x_k^T\}$  and using Assumptions 1(f) – (g) leads to

$$E\{\hat{x}_{k+1}\hat{x}_k^T\} = AE\{x_kx_k^T\} + E\{w_kx_k^T\}. \quad (96)$$

Substituting (94) - (96) into (89) gives

$$\hat{A} = A + E\{w_kx_k^T\}E\{\hat{x}_k\hat{x}_k^T\}^{-1}. \quad (97)$$

As explained in Section 8.5.2,  $x_k$  and  $w_k$  are uncorrelated, so (97) implies  $E\{\hat{A}\} = A$ .

The estimate (89) includes the correction term  $C^\dagger RC^{\dagger T}$  to accommodate noisy states. This generalises the least-squares estimate of  $A$  that is used within EM algorithms such as [20]. It is shown below that the estimate converges as the number of data points increases.

**Lemma 20:** The estimate (89) is asymptotically consistent.

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“It is unworthy of excellent men to lose hours like slaves in the labour of calculation which could be relegated to anyone else if machines were used.” *Gottfried Wilhelm von Leibnitz*

*Proof:* From (97) and Assumptions 1(f) – (g), it is found that

$$\begin{aligned} E\{(A - \hat{A})(A - \hat{A})^T\} &= E\{w_k x_k^T\} E\{x_k x_k^T\}^{-2} E\{x_k w_k^T\} \\ &= N^{-1} \sum_{k=1}^{\infty} w_k x_k^T E\{x_k x_k^T\}^{-2} N^{-1} \sum_{k=1}^{\infty} x_k w_k^T. \end{aligned}$$

since  $x_k$  and  $w_k$  are uncorrelated,  $\lim_{N \rightarrow \infty} E\{(A - \hat{A})(A - \hat{A})^T\} = 0$ .

The approach of [2] is used to show that (89) attains the corresponding CRLB. It follows from (87) that  $\hat{x}_{k+1} = x_{k+1} + C^\dagger v_{k+1} = Ax_k + u_k$ , where  $u_k = w_k + C^\dagger v_{k+1}$  and  $E\{u_k u_k^T\} = Q + C^\dagger R C^{\dagger T}$ . Suppose that  $\hat{x}_{k+1} \sim \mathcal{N}(Ax_k, Q + C^\dagger R C^{\dagger T})$ , i.e., the probability density function of  $\hat{x}_{k+1}$  is

$$\begin{aligned} p(\hat{x}_{k+1}) &= \frac{1}{(2\pi)^{|Q + C^\dagger R C^{\dagger T}|^{N/2}}} \\ &\times \exp\left\{-\frac{1}{2} \sum_{k=1}^N (\hat{x}_{k+1} - Ax_k)^T (Q + C^\dagger R C^{\dagger T})^{-1} (\hat{x}_{k+1} - Ax_k)\right\}. \end{aligned}$$

Setting  $\nabla_{\hat{A}} \log(p(\hat{x}_{k+1})) = 0$  leads to an MLE equal to (89). Straightforward algebraic manipulations result in

$$\nabla_{\hat{A}} \log(p(\hat{x}_{k+1})) = I(A)(\hat{A} - A), \quad (98)$$

where

$$I(A) = -\nabla_{\hat{A}}^2 \log(p(\hat{x}_{k+1})) = (Q + C^\dagger R C^{\dagger T})^{-1} \sum_{k=1}^n \hat{x}_k^2 - C^\dagger R C^{\dagger T}$$

is the corresponding Fisher information matrix, in which  $\nabla_{\hat{A}}^2$  denotes the Hessian with respect to  $\hat{A}$ . Since the estimate (89) satisfies (98), it attains the CRLB and is an efficient estimator [2].

### 8.5.3.3 Identification of Q with R Known

Unbiased estimates of  $Q$  are similarly determined below. Two cases are considered, depending on whether the actual or estimated  $A$  is available.

**Lemma 21:** Under Assumptions 1 - 2 with  $x_0 = 0$ :

(a)

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“Give a man a fish, and he will eat for a day. Give a man Twitter, and he will forget to eat and starve to death.” *Andy Borowitz*

$$P = \lim_{k \rightarrow \infty} E\{x_k x_k^T\}. \quad (99)$$

or equivalently

$$P = \lim_{k \rightarrow \infty} \Gamma_k Q \Gamma_k^T \quad (100)$$

satisfy

$$BQB^T = P - APA^T, \quad (101)$$

where  $\Gamma_k = \begin{bmatrix} I & A & \cdots & A^{k-1} \end{bmatrix}$  is the reachability matrix.

(b) For cases where the actual  $A$  is available, the estimate

$$\hat{Q} = E\{\hat{x}_k \hat{x}_k^T\} - C^\dagger RC^{\dagger T} - A(E\{\hat{x}_k \hat{x}_k^T\} - C^\dagger RC^{\dagger T})A^T \quad (102)$$

of  $Q$  is unbiased.

(c) For cases where the estimate  $\hat{A}$  of  $A$  is available, the estimate

$$\hat{Q} = E\{\hat{x}_k \hat{x}_k^T\} - C^\dagger RC^{\dagger T} - \hat{A}(E\{\hat{x}_k \hat{x}_k^T\} - C^\dagger RC^{\dagger T})\hat{A}^T \quad (103)$$

of  $Q$  is unbiased.

**Proof:** (a) Right-multiplying (81) by  $x_k$ , taking expectations and using (99) leads to (101). Equivalently, in a subspace identification framework [28], [14], with  $x_0$

$$= 0, \text{ note that } x_k = \Gamma_k \begin{bmatrix} w_{k-1} \\ \vdots \\ w_1 \\ w_0 \end{bmatrix}. \text{ Denote the } k\text{-step controllability gramian by } P_k =$$

$\Gamma_k Q \Gamma_k^T$ . The (steady state) controllability gramian is given by (20), which satisfies the Lyapunov equation (101).

(b) Substituting (14) in (22) results in

$$\hat{Q} = Q + E\{x_k v_k^T\} C^{\dagger T} + C^\dagger E\{v_k x_k^T\} - A(E\{x_k v_k^T\} C^{\dagger T} + C^\dagger E\{v_k x_k^T\}) A^T. \quad (104)$$

Since  $x_k$  and  $v_k$  are uncorrelated (from Assumption 1(g)), (24) implies  $E\{\hat{Q}\} = Q$ .

(c) Using (89) and (94) within (102) results in (103) from which the claim follows.

Thus,  $Q$  can be estimated by factorising a lower triangular matrix of the form (100) or more simply from the Lyapunov equation (101). It is noted below that (102) and (103) converge as the number of data points increases.

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“If it keeps up, man will atrophy all his limbs but the push-button finger.” Frank Lloyd Wright

**Lemma 22:** *The estimates (102) and (103) are asymptotically consistent.*

**Proof:** Since

$$\begin{aligned} & E\{(Q - \hat{Q})(Q - \hat{Q})^T\} \\ &= (E\{x_k v_k^T\} C^{\dagger T} + C^{\dagger} E\{v_k x_k^T\} - A(E\{x_k v_k^T\} C^{\dagger T} + C^{\dagger} E\{v_k x_k^T\}) A^T) \\ & \quad \times (E\{x_k v_k^T\} C^{\dagger T} + C^{\dagger} E\{v_k x_k^T\} - A(E\{x_k v_k^T\} C^{\dagger T} + C^{\dagger} E\{v_k x_k^T\}) A^T)^T \\ & \text{contains factors of the form } N^{-1} \sum_{k=1}^{\infty} C^{\dagger} v_k x_k^T \text{ and } N^{-1} \sum_{k=1}^{\infty} x_k v_k^T C^{\dagger T}, \\ & \lim_{N \rightarrow \infty} E\{(Q - \hat{Q})(Q - \hat{Q})^T\} = 0. \end{aligned}$$

It is similarly shown that (102) attains the corresponding CRLB. Suppose that  $\hat{x}_{k+1} \sim \mathcal{N}(Ax_k, BQB^T + C^{\dagger}RC^{\dagger T})$ , i.e., the probability density function of  $\hat{x}_{k+1}$  is

$$\begin{aligned} p(\hat{x}_{k+1}) &= \frac{1}{(2\pi)^{|Q + C^{\dagger}RC^{\dagger T}|}^{N/2}} \\ & \quad \times \exp\left\{-\frac{1}{2} \sum_{k=1}^N (\hat{x}_{k+1} - Ax_k)^T (Q + C^{\dagger}RC^{\dagger T})^{-1} (\hat{x}_{k+1} - Ax_k)\right\}. \end{aligned}$$

By setting  $\nabla_{BQB^T} \log(p(\hat{x}_{k+1})) = 0$ , it is easily confirmed that (102) is an MLE.

Straight forward manipulations reveal that

$$\nabla_{BQB^T}^2 \log(p(\hat{x}_{k+1})) = IQ(\hat{Q} - Q), \quad (105)$$

where

$$I(BQB^T) = -\nabla_{BQB^T}^2 \log(p(\hat{x}_{k+1})) = 0.5N(Q + C^{\dagger}R(C^{\dagger})^T)^{-2}$$

is the corresponding Fisher information matrix. Since  $\nabla_{BQB^T} \log(p(\hat{x}_{k+1}))$  is of the form (105), the estimate (102) attains the CRLB and is efficient [2].

### 8.5.3.4 Identification of R with A Known

suppose that  $A$  is known and its inverse exists. A rearrangement of (89) suggests that  $R$  can be identified as

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“If automobiles had followed the same development cycle as the computer, a Rolls-Royce would today cost \$100, get a million miles per gallon, and explode once a year, killing everyone inside.” *Mark Stephens*

$$\hat{R} = C(E\{\hat{x}_k \hat{x}_k^T\} - A^{-1}E\{\hat{x}_{k+1} \hat{x}_k^T\})C^T. \quad (106)$$

**Lemma 23:** Under Assumptions 1 - 2, the estimate (106) is (a) unbiased and (b) consistent.

**Proof:** (a) Substituting (94) and (96) into (106) yields

$$\hat{R} = R - CA^{-1}E\{w_k x_k^T\}C^T. \quad (107)$$

Since  $x_k$  is uncorrelated with  $w_k$ , (107) implies  $E\{\hat{R}\} = R$ .

(b) It follows from

$$E\{(R - \hat{R})(R - \hat{R})^T\} = \left( CA^{-1}N^{-1} \sum_{k=1}^{\infty} w_k x_k^T \right) \left( CA^{-1}N^{-1} \sum_{k=1}^{\infty} w_k x_k^T \right)^T.$$

that  $\lim_{N \rightarrow \infty} E\{(R - \hat{R})(R - \hat{R})^T\} = 0$ , since  $x_k$  is uncorrelated with  $w_k$ .

In order to investigate the CRLB for  $\hat{R}$ , suppose that  $\hat{x}_k \sim \mathcal{N}(x_k, C^\dagger R (C^\dagger)^T)$ , i.e., the probability density function of  $\hat{x}_k$  is

$$p(\hat{x}_k) = \frac{1}{(2\pi)^{|C^\dagger R C^{\dagger T}|^{N/2}}} \exp\left\{-\frac{1}{2} \sum_{k=1}^N (\hat{x}_k - x_k)^T (C^\dagger R C^{\dagger T})^{-1} (\hat{x}_k - x_k)\right\}.$$

Setting  $\nabla_{\hat{R}}^2 \log(p(\hat{x}_k)) = 0$  reveals that (106) is an MLE. It is easily verified that

$$\nabla_{\hat{R}}^2 \log(p(\hat{x}_k)) = I(R)(\hat{R} - R), \quad (108)$$

where  $I(R) = -\nabla_{\hat{R}}^2 \log(p(\hat{x}_k)) = 0.5NR^{-2}$  is the corresponding Fisher information matrix. Since the MLE satisfies (28), it attains the CRLB and is efficient [2].

### 8.5.3.5 Identification of unknown A, Q and R

The approaches of Sections 8.5.3.2 – 8.5.3.4 are now combined for the problem of estimating unknown  $A$ ,  $Q$ , and  $R$ . One way of proceeding is to search for the minima of the cost functions (90) for all admissible  $\hat{A}$  and  $\hat{R}$ , i.e.,

$$\{\hat{A}_i, \hat{R}\} = \arg \min_{\substack{A > 0, \quad |\hat{A}_i(\hat{A})| < 1, \quad \hat{R} > 0}} J_i(\hat{A}_i, C, \hat{R}, X_k, \hat{X}_{i,k+1}), \quad (109)$$

in which the relationship between  $X_k$  and  $\hat{X}_k$  is described by (91). From Lemmas 17 and 23, the search problem (109) has a minimum

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“What lies at the heart of every living thing is not a fire, not warm breath, not a ‘spark of life’. It is information, words, instructions.” *Clinton Richard Dawkins*

$J_i(\hat{A}_i, C, \hat{R}, X_k, \hat{X}_{i,k+1}) = 0$  at  $\hat{A} = A$  and  $\hat{R} = R$ . After  $A$  and  $R$  have been identified from (109), the unknown  $\hat{Q}$  can be obtained (103).

#### 8.5.4 Example

Model outputs (2) of length  $N = 1,000,000$  were synthesised using  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0 & 0.7 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  and Gaussian noise realisations.

Measurements were then generated using independent Gaussian noise realisations within (3), and the signal-to-noise ratio (SNR) was varied in 1 dB steps, from -5 to 5 dB.

Unbiased estimates  $\hat{A} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix}$  and  $\hat{R}$  were obtained from the measurements by searching for the minimum of (109). The search space included 81 candidate  $\hat{A}$  matrices, namely,  $\hat{a}_{ij} = \{a_{ij} - 0.1, a_{ij}, a_{ij} + 0.1\}$ ,  $i, j = \{1, 2\}$ , and 11 candidate  $\hat{R}$  matrices, corresponding to the afore-mentioned SNRs. An unbiased estimate  $\hat{Q}$  was then obtained using the identified  $\hat{A}$  and  $\hat{R}$  within (103). Conventional (biased) estimates of  $A$  and  $Q$  were also calculated by neglecting  $R$  within (89) and (103), respectively.

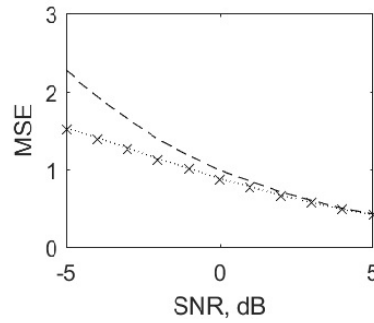


Fig. 7. Mean square error exhibited by filters designed with: actual  $A$ ,  $Q$  and  $R$  (crossed line); unbiased estimates of  $A$ ,  $Q$  and  $R$  (dotted line); and biased estimates of  $A$ ,  $Q$  and unbiased estimate of  $R$  (dashed line).

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“In my lifetime, we’ve gone from Eisenhower to George W. Bush. We’ve gone from John F. Kennedy to Al Gore. If this is evolution, I believe that in twelve years, we’ll be voting for plants.” *Lewis Niles Black*

An optimal Kalman filter was designed using the actual  $A$ ,  $Q$  and  $R$ . The MSEs observed for the filter operating on the measurements are indicated by the crossed line of Fig. 7. A second filter was designed using the unbiased estimates of  $A$ ,  $Q$  and  $R$ , for which the resulting MSEs are indicated by the dotted line of Fig. 7. It can be seen that the filters designed with actual and unbiased parameter estimates exhibit indistinguishable performance, which illustrates the asymptotic consistency results of Lemmas 19, 22 and 23.

A third filter was designed using the conventional (biased) least-squares estimates of  $A$  and  $Q$ , plus the unbiased estimate of  $R$ , for which the resulting MSEs are indicated by the dashed line of Fig. 1. It can be seen that this filter exhibits the worst performance. As expected, retaining the correction term,  $C^\dagger R C^{\dagger T}$ , within (89) and (103) returns a negligible benefit at high SNR.

## 8.6 Chapter Summary

Optimal predictor, filter and smoother designs require knowledge of model parameters. The least-squares solutions for estimating  $A$  or  $Q$  from the data are well known. From the Central Limit Theorem, the mean of a large sample of independent identically distributed random variables asymptotically approaches a normal distribution. Consequently, parameter estimates are often obtained by maximising Gaussian log-likelihood functions. Indeed, the MLEs of  $A$  or  $Q$  are the same as the least-squares estimates, which listed in Table 1. They are unbiased provided that the assumed models are correct, perfect (or noiseless) state information is available and the number of samples is sufficiently large.

Usually, both states and parameters need to be estimated from noisy measurements. The EM algorithm is a common technique for solving joint state and parameter estimation problems. It has been shown that the estimation sequences vary monotonically and depend on the initial conditions. An examination of the approximate Cramér-Rao lower bounds shows that the use of smoothed states as opposed to filtered states is expected to provide improved parameter estimation accuracy.

When the SNR is sufficiently high, the states can be recovered exactly and the bias errors diminish to zero, in which case  $\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0} \hat{Q} = Q$  and  $\lim_{Q^{-1} \rightarrow 0, R \rightarrow 0} \hat{A} = A$ .

Therefore, the process noise covariance and state matrix estimation procedures described herein are advocated when the measurement noise is negligible. Conversely, when the SNR is sufficiently low, that is, when the estimation problem is dominated by measurement noise, then  $\lim_{Q \rightarrow 0, R^{-1} \rightarrow 0} \hat{R} = R$ . This suggests

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“It took less than an hour to make the atoms, a few hundred million years to make the stars and planets, but five billion years to make man!” *George Gamow*

that measurement noise covariance estimates are best calculated when the signal is absent.

The afore-mentioned high-SNR asymptotes prompt the inclusion of a correction term for problems where imperfect or noisy state estimates are available. The resulting MLEs for  $A$ ,  $Q$  and  $R$  are listed in Table 2, which are unbiased provided that the assumed models are correct and the number of samples is sufficiently large. It is established that these estimates are asymptotically consistent. It is also shown under prescribed conditions that they are MLEs which attain the corresponding CRLBs.

	ASSUMPTIONS	MAIN RESULTS
Process Noise covariance	$x_{k+1} = Ax_k + Bw_k$ $z_k = Cx_k + v_k$	$\hat{Q} = E\{x_k x_k^T\} - A(E\{x_k x_k^T\})A^T$
State matrix		$\hat{A} = E\{x_{k+1} x_k^T\} E\{x_k x_k^T\}^{-1}$
Measurement noise covariance		$\hat{R} = C(E\{x_k x_k^T\} - A^{-1}E\{x_{k+1} x_k^T\})C^T$

Table 1. Estimates of process noise covariance, state matrix and measurement noise variance, assuming that the actual states are available.

	ASSUMPTIONS	MAIN RESULTS
Process noise covariance	$x_{k+1} = Ax_k + Bw_k$ $z_k = Cx_k + v_k$ $\hat{x}_k = C^\dagger z_k$	$\hat{Q} = E\{\hat{x}_k \hat{x}_k^T\} - C^\dagger RC^{\dagger T} - A(E\{\hat{x}_k \hat{x}_k^T\} - C^\dagger RC^{\dagger T})A^T$
State matrix		$\hat{A} = E\{\hat{x}_{k+1} \hat{x}_k^T\} (E\{\hat{x}_k \hat{x}_k^T\} - C^\dagger RC^{\dagger T})^{-1}$
Measurement noise covariance		$\hat{R} = C(E\{\hat{x}_k \hat{x}_k^T\} - A^{-1}E\{\hat{x}_{k+1} \hat{x}_k^T\})C^T$

Table 2. Estimates of process noise variance, state matrix element and measurement noise variance, assuming that estimated states are available.

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“The faithful duplication and repair exhibited by the double-stranded DNA structure would seem to be incompatible with the process of evolution. Thus, evolution has been explained by the occurrence of errors during DNA replication and repair.” *Tomoyuki Shibata*



In the event that  $A$ ,  $Q$  and  $R$  are all unknown, a search may be conducted for the minimum of cost functions for all admissible  $A$  and  $R$  (as described above). The MLE for  $Q$  may be calculated subsequently. It is demonstrated with the aid of a modelling study that a filter designed with the resulting unbiased parameters can outperform a filter designed with conventional parameter estimates.

In summary, for cases where perfect or noiseless states are available, the standard least squares or MLEs (shown in Table 1) should be satisfactory. If the states are accessible and moderate measurement noise is present, the unbiased MLEs (shown in Table 2) can be used. Otherwise designers may have to consider using EM algorithms for joint state and parameter estimation.

## 8.7 Problems

### Problem 1.

- (i) Consider the second order difference equation  $x_{k+2} + a_1x_{k+1} + a_0x_k = w_k$ . Assuming that  $w_k \sim \mathcal{N}(0, \sigma_w^2)$ , obtain an equation for the MLEs of the unknown  $a_1$  and  $a_0$ .
- (ii) Consider the  $n^{\text{th}}$  order autoregressive system  $x_{k+n} + a_{n-1}x_{k+n-1} + a_{n-2}x_{k+n-2} + \dots + a_0x_k = w_k$ , where  $a_{n-1}, a_{n-2}, \dots, a_0$  are unknown. From the assumption  $w_k \sim \mathcal{N}(0, \sigma_w^2)$ , obtain an equation for MLEs of the unknown coefficients.

**Problem 2.** Suppose that  $N$  samples of  $x_{k+1} = Ax_k + w_k$  are available, where  $w_k \sim \mathcal{N}(0, \sigma_w^2)$ , in which  $\sigma_w^2$  is an unknown parameter.

- (i) Write down a Gaussian log-likelihood function for the unknown parameter, given  $x_k$ .
- (ii) Derive a formula for the MLE  $\hat{\sigma}_w^2$  of  $\sigma_w^2$ .
- (iii) Show that  $E\{\hat{\sigma}_w^2\} = \sigma_w^2$  provided that  $N$  is large.
- (iv) Find the Cramér Rao lower bound for  $\hat{\sigma}_w^2$ .
- (v) Replace the actual states  $x_k$  with filtered state  $\hat{x}_{k/k}$  within the MLE formula. Obtain a high SNR asymptote for this approximate MLE.

**Problem 3.** Consider the state evolution  $x_{k+1} = Ax_k + w_k$ , where  $A \in \mathbb{R}^{n \times n}$  is unknown and  $w_k \in \mathbb{R}^n$ .

- (i) Write down a Gaussian log-likelihood function for the unknown components  $a_{ij}$  of  $A$ , given  $x_k$  and  $x_{k+1}$ .
- (ii) Derive a formula for the MLE of  $a_{ij}$ .
- (iii) Show that  $\hat{a}_{ij} = a_{ij}$ . Replace the actual states  $x_k$  with the filtered state within the obtained formula to yield an approximate MLE for  $a_{ij}$ .

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“It took less than an hour to make the atoms, a few hundred million years to make the stars and planets, but five billion years to make man!” *George Gamow*

(iv) Obtain a high SNR asymptote for the approximate MLE.

**Problem 4.** Consider measurements of a sinusoidal signal modelled by  $y_k = A \cos(2\pi f k + \phi) + v_k$ , with amplitude  $A > 0$ , frequency  $0 < f < 0.5$ , phase  $\phi$  and Gaussian white measurement noise  $v_k$ .

- (i) Assuming that  $\phi$  and  $f$  are known, determine the Fisher information and the Cramér Rao lower bound for an unknown  $A$ .
- (ii) Assuming that  $A$  and  $\phi$  are known, determine the Fisher information and the Cramér Rao lower bound for an unknown  $f_0$ .
- (iii) Assuming that  $A$  and  $f$  are known, determine the Fisher information and the Cramér Rao lower bound.
- (iv) Assuming that the vector parameter  $[A^T, \phi^T]^T$  is known, determine the Fisher information matrix and the Cramér Rao lower bound. (Hint: use small angle approximations for sine and cosine, see [2].)

## 8.8 Glossary

SNR	Signal to noise ratio.
MLE	Maximum likelihood estimate.
CRLB	Cramér Rao Lower Bound
$F(\theta)$	The Fisher information of a parameter $\theta$ .
$x_k \sim \mu, \sigma^2$	The random variable $x_k$ is normally distributed with mean $\mu$ and variance $\sigma^2$ .
$w_{i,k}, v_{i,k}, z_{i,k}$	$i^{\text{th}}$ elements of vectors $w_k, v_k, z_k$ .
$\sigma_{i,k}^2$	Estimates of variances of $w_{i,k}$ and $v_{i,k}$ at iteration $u$ .
$\hat{A}_u, \hat{R}_u, \hat{Q}_u$	Estimates of state matrix $A$ , covariances $R$ and $Q$ at iteration $u$ .
$\lambda_i$	The $i$ eigenvalues of $A$ .
$A_i, C_i$	$i^{\text{th}}$ row of state-space matrices $A$ and $C$ .
$K_{i,k}, L_{i,k}$	$i^{\text{th}}$ row of predictor and filter gain matrices $K_k$ and $L_k$ .
$\epsilon_k$	Additive term within the design Riccati difference equation to account for the presence of modelling error at time $k$ and iteration $u$ .
$a_{i,j}$	Element in row $i$ and column $j$ of $A$ .
$C_k^+$	Moore-Penrose pseudo-inverse of $C_k$ .
$\mathcal{E}_i$	A system (or map) that operates on the estimation problem inputs $i$ and produces the output estimation error at iteration $u$ . It is convenient to make use of the factorisation $\mathcal{E}_i = \mathcal{E}_i^+ + \mathcal{E}_i^-$ , where $\mathcal{E}_i^+$ depends the filter or smoother solution and $\mathcal{E}_i^-$ is a lower performance bound.

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“We don’t know all the answers. If we knew all the answers we’d get bored, wouldn’t we? We keep looking, searching, trying to get more knowledge.” *Jack LaLanne*

## 8.9 References

- [1] L. L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time Series Analysis*, Addison-Wesley Publishing Company Inc., Massachusetts, USA, 1990.
- [2] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice Hall, Englewood Cliffs, New Jersey, ch. 7, pp. 157 – 204, 1993.
- [3] G. J. McLachlan and T. Krishnan, *The EM Algorithm and Extensions*, John Wiley & Sons, Inc., New York, 1997.
- [4] H. L. Van Trees and K. L. Bell (Editors), *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*, John Wiley & Sons, Inc., New Jersey, 2007.
- [5] A. Van Den Bos, *Parameter Estimation for Scientists and Engineers*, John Wiley & Sons, New Jersey, 2007.
- [6] R. K. Mehra, “On the identification of variances and adaptive Kalman filtering”, *IEEE Transactions on Automatic Control*, vol. 15, pp. 175 – 184, Apr. 1970.
- [7] D. C. Rife and R. R. Boorstyn, “Single-Tone Parameter Estimation from Discrete-time Observations”, *IEEE Transactions on Information Theory*, vol. 20, no. 5, pp. 591 – 598, Sep. 1974.
- [8] R. P. Nayak and E. C. Foundriat, “Sequential Parameter Estimation Using Pseudoinverse”, *IEEE Transactions on Automatic Control*, vol. 19, no. 1, pp. 81 – 83, Feb. 1974.
- [9] P. R. Bélanger, “Estimation of Noise Covariance Matrices for a Linear Time-Varying Stochastic Process”, *Automatica*, vol. 10, pp. 267 – 275, 1974.
- [10] V. Strejc, “Least Squares Parameter Estimation”, *Automatica*, vol. 16, pp. 535 – 550, Sep. 1980.
- [11] A. P. Dempster, N. M. Laird and D. B. Rubin, “Maximum likelihood from incomplete data via the EM algorithm,” *Journal of the Royal Statistical Society*, vol 39, no. 1, pp. 1 – 38, 1977.
- [12] P. Van Overschee and B. De Moor, “A Unifying Theorem for Three Subspace System Identification Algorithms”, *Automatica*, 1995.
- [13] T. Katayama and G. Picci, “Realization of stochastic systems with exogenous inputs and subspace identification methods”, *Automatica*, vol. 35, pp. 1635 – 1652, 1999.
- [14] T. Katayama, *Subspace Methods for System Identification*, Springer-Verlag London Limited, 2005.
- [15] R. H. Shumway and D. S. Stoffer, “An approach to time series smoothing and forecasting using the EM algorithm,” *Journal of Time Series Analysis*, vol. 3, no. 4, pp. 253 – 264, 1982.
- [16] C. F. J. Wu, “On the convergence properties of the EM algorithm,” *Annals of Statistics*, vol. 11, no. 1, pp. 95 – 103, Mar. 1983.

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“There is nothing more practical than a good theory.” *Leonid Ilich Brezhnev*

- [17] M. Feder and E. Weinstein, "Parameter estimation of superimposed signals using the EM algorithm," *IEEE Transactions on Signal Processing*, vol. 36, no. 4, pp. 477 – 489, Apr. 1988.
- [18] G. A. Einicke, "Optimal and Robust Noncausal Filter Formulations", *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 1069 - 1077, Mar. 2006.
- [19] G. A. Einicke, J. T. Malos, D. C. Reid and D. W. Hainsworth, "Riccati Equation and EM Algorithm Convergence for Inertial Navigation Alignment", *IEEE Transactions on Signal Processing*, vol 57, no. 1, Jan. 2009.
- [20] G. A. Einicke, G. Falco and J. T. Malos, "EM Algorithm State Matrix Estimation for Navigation", *IEEE Signal Processing Letters*, vol. 17, no. 5, pp. 437 – 440, May 2010.
- [21] T. K. Moon, "The Expectation-Maximization Algorithm", *IEEE Signal Processing Magazine*, vol. 13, pp. 47 – 60, Nov. 1996.
- [22] D. G. Tzikas, A. C. Likas and N. P. Galatsanos, "The Variational Approximation for Bayesian Inference: Life After the EM Algorithm", *IEEE Signal Processing Magazine*, vol. 25, Iss. 6, pp. 131 – 146, Nov. 2008.
- [23] D. M. Titterton, A. F. M. Smith and U. E. Makov, *Statistical Analysis of Finite Mixture Distributions*, Wiley, Chichester and New York, 1985.
- [24] R. P. Savage, *Strapdown Analytics*, Strapdown Associates, vol. 2, ch. 15, pp. 15.1 – 15.142, 2000.
- [25] P. K. Seidelmann, ed., *Explanatory supplement to the Astronomical Almanac*, Mill Valley, Cal., University Science Books, pp. 52 and 698, 1992.
- [26] W. E. Larimore, "Maximum Likelihood Subspace Identification for Linear, Nonlinear and Closed-loop Systems", *Proc. American Control Conf.*, pp. 2305 – 2319, Jun. 2005.
- [27] F. Ding, L. Qiu and T. Chen, "Reconstruction of continuous-time systems from their non-uniformly sampled discrete-time systems", *Automatica*, 2009.
- [28] H. Hjalmarsson and B. Ninness, "Fast, non-iterative estimation of hidden Markov models", *Proc. IEEE Conf. Acoustics, Speech and Signal Processing (ICASSP'98)*, pp. 2253 - 2256, 1998.
- [29] W. X. Zheng, "On a Least-Squares Based Algorithm for Identification of Stochastic Linear Systems", *IEEE Trans. Sig. Process.*, vol. 46, no. 6, pp. 1631 – 1638, Jun. 1998.
- [30] W. X. Zheng, "A Least-Squares Based Method for Autoregressive Signals in the Presence of Noise", *IEEE Trans. Circuits and Systems II*, vol. 46, no. 1, pp. 81 – 85, Jan. 1999.

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"Wastepaper-baskets were containers used in the seventeenth century for the disposal of some first versions of manuscripts which self-criticism - or private criticism of learned friends - ruled out on the first reading. In our age of publication explosion most people have no time to read their manuscripts, and the function of wastepaper-baskets has now been taken over by scientific journals." *Imre Lakatos, The methodology of scientific research programmes*

- [31] Y. Xia and W. X. Zheng, “On unbiased identification of autoregressive signals with noisy measurements”, *Proc. IEEE Int. Symp. Circuits and Systems*, Lisbon, pp. 2157 – 2160, May 2015.
- [32] F. Ding, T. Chen and L. Qiu, “Bias Compensation Recursive Least-Squares Identification Algorithm for MISO Systems”, *IEEE Trans. Circuits and Systems II*, vol. 53, no. 5, pp. 349 - 353, May 2006.
- [33] Y. Zhang, “Unbiased identification of a class of multi-input single-output systems with correlated disturbances using bias compensation methods”, *Mathematical and Computer Modelling*, vol. 53, pp. 1810 – 1819, 2011.
- [34] J. C. A. Barata and M. S. Hussein "The Moore-Penrose Pseudoinverse: A Tutorial Review of the Theory", *Brazilian J. of Physics*, vol. 42, iss. 1, pp. 146 -165, 2012.

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“The idea is to try to give all the information to help others to judge the value of your contribution; not just the information that leads to judgment in one particular direction or another.” *Richard Phillips Feynman*

## 9. Robust Prediction, Filtering and Smoothing

### 9.1 Introduction

The previously-discussed optimum predictor, filter and smoother solutions assume that the model parameters are correct, the noise processes are white and their associated covariances are known precisely. These solutions are optimal in a mean-square-error sense, that is they provide the best average performance. If the above assumptions are correct, then the filter's mean-square-error equals the trace of design error covariance. The underlying modelling and noise assumptions are a often convenient fiction. They do, however, serve to allow estimated performance to be weighed against implementation complexity.

In general, robustness means “the persistence of a system's characteristic behaviour under perturbations or conditions of uncertainty” [1]. In an estimation context, robust solutions refer to those that accommodate uncertainties in problem specifications. They are also known as worst-case or peak error designs. The standard predictor, filter and smoother structures are retained but a larger design error covariance is used to account for the presence of modelling error.

Designs that cater for worst cases are likely to exhibit poor average performance. Suppose that a bridge designed for average loading conditions returns an acceptable cost benefit. Then a robust design that is focussed on accommodating infrequent peak loads is likely to provide worse cost performance. Similarly, a worst-case shoe design that accommodates rarely occurring large feet would provide poor fitting performance on average. That is, robust designs tend to be conservative. In practice, a trade-off may be desired between optimum and robust designs.

The material canvassed herein is based on the  $H_\infty$  filtering results from robust control. The robust control literature is vast, see [2] – [33] and the references therein. As suggested above, the  $H_\infty$  solutions of interest here involve observers having gains that are obtained by solving Riccati equations. This Riccati equation solution approach relies on the Bounded Real Lemma – see the pioneering work

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“On a huge hill, Cragged, and steep, Truth stands, and he that will Reach her, about must, and about must go.” *John Donne*

by Vaidyanathan [2] and Petersen [3]. The Bounded Real Lemma is implicit with game theory [9] – [19]. Indeed, the continuous-time solutions presented in this section originate from the game theoretic approach of Doyle, Glover, Khargonekar, Francis Limebeer, Anderson, Khargonekar, Green, Theodore and Shaked, see [4], [13], [15], [21]. The discussed discrete-time versions stem from the results of Limebeer, Green, Walker, Yaesh, Shaked, Xie, de Souza and Wang, see [5], [11], [18], [19], [21]. In the parlance of game theory: “a statistician is trying to best estimate a linear combination of the states of a system that is driven by nature; nature is trying to cause the statistician’s estimate to be as erroneous as possible, while trying to minimise the energy it invests in driving the system” [19].

Pertinent state-space  $H_\infty$  predictors, filters and smoothers are described in [4] – [19]. Some prediction, filtering and smoothing results are summarised in [13] and methods for accommodating model uncertainty are described in [14], [18], [19]. The aforementioned methods for handling model uncertainty can result in conservative designs (that depart far from optimality). This has prompted the use of linear matrix inequality solvers in [20], [23] to search for optimal solutions to model uncertainty problems.

It is explained in [15], [19], [21] that a saddle-point strategy for the games leads to robust estimators, and the resulting robust smoothing, filtering and prediction solutions are summarised below. While the solution structures remain unchanged, designers need to tweak the scalar within the underlying Riccati equations.

This chapter has two main parts. Section 9.2 describes robust continuous-time solutions and the discrete-time counterparts are presented in Section 9.3. The previously discussed techniques each rely on a trick. The optimum filters and smoothers arise by completing the square. In maximum-likelihood estimation, a function is differentiated with respect to an unknown parameter and then set to zero. The trick behind the described robust estimation techniques is the Bounded Real Lemma, which opens the discussions.

## 9.2 Robust Continuous-time Estimation

### 9.2.3 Continuous-Time Bounded Real Lemma

First, consider the unforced system

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

---

“Uncertainty is one of the defining features of Science. Absolute proof only exists in mathematics. In the real world, it is impossible to prove that theories are right in every circumstance; we can only prove that they are wrong. This provisionality can cause people to lose faith in the conclusions of science, but it shouldn’t. The recent history of science is not one of well-established theories being proven wrong. Rather, it is of theories being gradually refined.” *New Scientist* vol. 212 no. 2835

over a time interval  $t \in [0, T]$ , where  $A(t) \in \mathbb{R}^{n \times n}$ . For notational convenience, define the stacked vector  $x = \{x(t), t \in [0, T]\}$ . From Lyapunov stability theory [36], the system (1) is asymptotically stable if there exists a function  $V(x(t)) > 0$  such that  $\dot{V}(x(t)) < 0$ . A possible Lyapunov function is  $V(x(t)) = x^T(t)P(t)x(t)$ , where  $P(t) = P^T(t) \in \mathbb{R}^{n \times n}$  is positive definite. To ensure  $x \in \mathcal{L}_2$  it is required to establish that

$$\dot{V}(x(t)) = \dot{x}^T(t)P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)\dot{x}(t) < 0. \quad (2)$$

Now consider the output of a linear time varying system,  $y = \mathcal{G}w$ , having the state-space representation

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (3)$$

$$y(t) = C(t)x(t), \quad (4)$$

where  $w(t) \in \mathbb{R}^m$ ,  $B(t) \in \mathbb{R}^{n \times m}$  and  $C(t) \in \mathbb{R}^{p \times n}$ . Assume temporarily that  $E\{w(t)w^T(\tau)\} = I\delta(t-\tau)$ . The Bounded Real Lemma [13], [15], [21], states that  $w \in \mathcal{L}_2$  implies  $y \in \mathcal{L}_2$  if

$$\dot{V}(x(t)) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) < 0 \quad (5)$$

for a  $\gamma \in \mathbb{R}$ . Integrating (5) from  $t = 0$  to  $t = T$  gives

$$\int_0^T \dot{V}(x(t)) dt + \int_0^T y^T(t)y(t) dt - \gamma^2 \int_0^T w^T(t)w(t) dt < 0 \quad (6)$$

and noting that  $\int_0^T \dot{V}(x(t)) dt = x^T(T)P(T)x(T) - x^T(0)P(0)x(0)$ , another objective is

$$\frac{x^T(T)P(T)x(T) - x^T(0)P(0)x(0) + \int_0^T y^T(t)y(t) dt}{\int_0^T w^T(t)w(t) dt} \leq \gamma^2. \quad (7)$$

Under the assumptions  $x(0) = 0$  and  $P(T) = 0$ , the above inequality simplifies to

$$\frac{\|y(t)\|_2^2}{\|w(t)\|_2^2} = \frac{\int_0^T y^T(t)y(t) dt}{\int_0^T w^T(t)w(t) dt} \leq \gamma^2. \quad (8)$$

The  $\infty$ -norm of  $\mathcal{G}$  is defined as

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“Information is the resolution of uncertainty.” *Claude Elwood Shannon*



$$\|\mathcal{G}\|_\infty = \frac{\|y\|_2}{\|w\|_2} = \frac{\|\mathcal{G}w\|_2}{\|w\|_2}. \quad (9)$$

The Lebesgue  $\infty$ -space is the set of systems having finite  $\infty$ -norm and is denoted by  $\mathcal{L}_\infty$ . That is,  $\mathcal{G} \in \mathcal{L}_\infty$ , if there exists a  $\gamma \in \mathbb{R}$  such that

$$\sup_{\|w\|_2 \neq 0} \|\mathcal{G}\|_\infty = \sup_{\|w\|_2 \neq 0} \frac{\|y\|_2}{\|w\|_2} \leq \gamma, \quad (10)$$

namely, the supremum (or maximum) ratio of the output and input 2-norms is finite. The conditions under which  $\mathcal{G} \in \mathcal{L}_\infty$  are specified below. The accompanying sufficiency proof combines the approaches of [15], [31]. A further five proofs for this important result appear in [21].

**Lemma 1:** *The continuous-time Bounded Real Lemma [15], [13], [21]: In respect of the above system  $\mathcal{G}$ , suppose that the Riccati differential equation*

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + C^T(t)C(t) + \gamma^{-2}P(t)B(t)B^T(t)P(t), \quad (11)$$

*has a solution on  $[0, T]$ . Then  $\|\mathcal{G}\|_\infty \leq \gamma$  for any  $w \in \mathcal{L}_2$ .*

**Proof:** From (2) – (5),

$$\begin{aligned} \dot{V}(t) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) &= x^T(t)C^T(t)C(t)x(t) - \gamma^2 w^T(t)w(t) + x^T(t)\dot{P}(t)x(t) \\ &\quad + (A(t)x(t) + B(t)w(t))^T P(t)x(t) + x^T(t)P(t)(A(t)x(t) + B(t)w(t)) \\ &= \gamma^{-2}x^T(t)P(t)B(t)B^T(t)P(t)x(t) - \gamma^2 w^T(t)w(t) \\ &\quad + w^T(t)B^T(t)P(t)x(t) + x^T(t)P(t)B(t)w(t) \\ &= -\gamma^2(w(t) - \gamma^{-2}B(t)P(t)x(t))^T(w(t) - \gamma^{-2}B(t)P(t)x(t)), \end{aligned}$$

*which implies (6) and (7). Inequality (8) is established under the assumptions  $x(0) = 0$  and  $P(T) = 0$ .*

In general, where  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau)$ , the scaled matrix  $\bar{B}(t) = B(t)Q^{1/2}(t)$  may be used in place of  $B(t)$  above. When the plant  $\mathcal{G}$  has a direct feedthrough matrix, that is,

$$y(t) = C(t)x(t) + D(t)w(t), \quad (12)$$

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“All exact science is dominated by the idea of approximation.” *Earl Bertrand Arthur William*

$D(t) \in \mathbb{R}^{p \times m}$ , the above Riccati differential equation is generalised to

$$-\dot{P}_1(t) = P_1(t)(A(t) + B(t)M^{-1}(t)D^T(t)C(t)) + (A(t) + B(t)M^{-1}(t)D^T(t)C(t))^T P_1(t) + \gamma^{-2}B(t)M^{-1}(t)B^T(t) + C^T(t)(I + D(t)M^{-1}(t)D^T(t))C(t), \quad (13)$$

where  $M(t) = \gamma^2 I - D^T(t)D(t) > 0$ . A proof is requested in the problems.

Criterion (8) indicates that the ratio of the system's output and input energies is bounded above by  $\gamma^2$  for any  $w \in \mathcal{L}_2$ , including worst-case  $w$ . Consequently, solutions satisfying (8) are often called worst-case designs.

## 9.2.4 Continuous-Time $H_\infty$ Filtering

### 9.2.4.1 Problem Definition

Now that the Bounded Real Lemma has been defined, the  $H_\infty$  filter can be set out. The general filtering problem is depicted in Fig. 1. It is assumed that the system  $\mathcal{G}_2$  has the state-space realisation

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad x(0) = 0, \quad (14)$$

$$y_2(t) = C_2(t)x(t). \quad (15)$$

Suppose that the system  $\mathcal{G}_1$  has the realisation (14) and

$$y_1(t) = C_1(t)x(t). \quad (16)$$

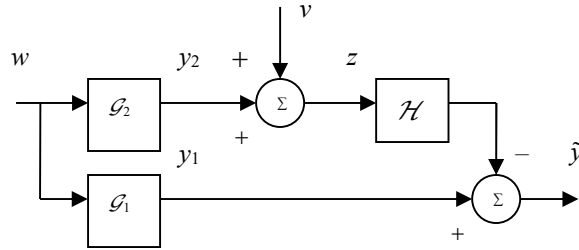


Fig. 1. The general filtering problem. The objective is to estimate the output of  $\mathcal{G}_1$  from noisy measurements of  $\mathcal{G}_2$ .

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“Although economists have studied the sensitivity of import and export volumes to changes in the exchange rate, there is still much uncertainty about just how much the dollar must change to bring about any given reduction in our trade deficit.” *Martin Stuart Feldstein*

It is desired to find a causal solution  $\mathcal{H}$  that produces estimates  $\hat{y}_1(t|t)$  of  $y_1(t)$  from the measurements,

$$z(t) = y_2(t) + v(t), \quad (17)$$

at time  $t$  so that the output estimation error,

$$\tilde{y}(t|t) = y_1(t) - \hat{y}_1(t|t), \quad (18)$$

is in  $\mathcal{L}_2$ . The error signal (18) is generated by a system denoted by  $\tilde{y} = \mathcal{R}_{yi}i$ ,

where  $i = \begin{bmatrix} v \\ w \end{bmatrix}$  and  $\mathcal{R}_{yi} = -[\mathcal{H} \quad \mathcal{H}\mathcal{G}_2 - \mathcal{G}_1]$ . Hence, the objective is to achieve

$\int_0^T \tilde{y}^T(t|t)\tilde{y}(t|t) dt - \gamma^2 \int_0^T i^T(t)i(t) dt < 0$  for some  $\gamma \in \mathbb{R}$ . For convenience, it is assumed here that  $w(t) \in \mathbb{R}^m$ ,  $E\{w(t)\} = 0$ ,  $E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau)$ ,  $v(t) \in \mathbb{R}^p$ ,  $E\{v(t)\} = 0$ ,  $E\{v(t)v^T(\tau)\} = R(t)\delta(t-\tau)$  and  $E\{w(t)v^T(\tau)\} = 0$ .

#### 9.2.4.2 $H_\infty$ Solution

A parameterisation of all solutions for the  $H_\infty$  filter is developed in [21]. A minimum-entropy filter arises when the contractive operator within [21] is zero and is given by

$$\dot{\hat{x}}(t|t) = (A(t) - K(t)C_2(t))\hat{x}(t|t) + K(t)z(t), \quad \hat{x}(0) = 0, \quad (19)$$

$$\hat{y}_1(t|t) = C_1(t)\hat{x}(t|t), \quad (20)$$

where

$$K(t) = P(t)C_2^T(t)R^{-1}(t) \quad (21)$$

is the filter gain and  $P(t) = P^T(t) > 0$  is the solution of the Riccati differential equation

$$\begin{aligned} \dot{P}(t) = & A(t)P(t) + P(t)A^T(t) + B(t)Q(t)B^T(t) \\ & - P(t)(C_2^T(t)R^{-1}(t)C_2(t) - \gamma^{-2}C_1^T(t)C_1(t))P(t), \quad P(0) = 0. \end{aligned} \quad (22)$$

It can be seen that the  $H_\infty$  filter has a structure akin to the Kalman filter. A point of difference is that the solution to the above Riccati differential equation solution depends on  $C_1(t)$ , the linear combination of states being estimated.

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“Uncertainty and expectation are the joys of life. Security is an insipid thing, through the overtaking and possessing of a wish discovers the folly of the chase.” *William Congreve*

### 9.2.4.3 Properties

Define  $\bar{A}(t) = A(t) - K(t)C_2(t)$ . Subtracting (19) – (20) from (14) – (15) yields the error system

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}(t|t) \\ \tilde{y}(t|t) \end{bmatrix} &= \begin{bmatrix} \bar{A}(t) & [-K(t) & B(t)] \\ C_1(t) & [0 & 0] \end{bmatrix} \begin{bmatrix} \tilde{x}(t|t) \\ v(t) \\ w(t) \end{bmatrix}, \quad \tilde{x}(0) = 0, \\ &= \mathcal{R}_{yi} i, \end{aligned} \quad (23)$$

where  $\tilde{x}(t|t) = x(t) - \hat{x}(t|t)$  and  $\mathcal{R}_{yi} = \begin{bmatrix} \bar{A}(t) & [-K(t) & B(t)] \\ C_1(t) & [0 & 0] \end{bmatrix}$ . The adjoint of

$\mathcal{R}_{yi}$  is given by

$$\mathcal{R}_{yi}^H = \begin{bmatrix} -\bar{A}^T(t) & C_1^T(t) \\ K^T(t) & [0] \\ -B^T(t) & [0] \end{bmatrix}. \text{ It is shown below that the estimation error satisfies}$$

the desired performance objective.

**Lemma 2:** *In respect of the  $H_\infty$  problem (14) – (18), the solution (19) – (20) achieves the performance  $x^T(T)P(T)x(T) - x^T(0)P(0)x(0) + \int_0^T \tilde{y}^T(t|t)\tilde{y}(t|t) dt - \gamma^2 \int_0^T i^T(t)i(t) dt < 0$ .*

**Proof:** *Following the approach in [15], [21], by applying Lemma 1 to the adjoint of (23), it is required that there exists a positive definite symmetric solution to*

$$\begin{aligned} -\dot{P}(\tau) &= \bar{A}(\tau)P(\tau) + P(\tau)\bar{A}^T(\tau) + B(\tau)Q(\tau)B^T(\tau) + K(\tau)R(\tau)K^T(\tau) \\ &\quad + \gamma^{-2}P(\tau)C_1^T(\tau)C_1(\tau)P(\tau), \quad P(\tau)|_{\tau=T} = 0, \end{aligned}$$

on  $[0, T]$  for some  $\gamma \in \mathbb{R}$ , in which  $\tau = T - t$  is a time-to-go variable. Substituting  $K(\tau) = P(\tau)C_2^T(\tau)R^{-1}(\tau)$  into the above Riccati differential equation yields

$$\begin{aligned} -\dot{P}(\tau) &= A(\tau)P(\tau) + P(\tau)A^T(\tau) + B(\tau)Q(\tau)B^T(\tau) \\ &\quad - P(\tau)(C_2^T(\tau)R^{-1}(\tau)(C_2(\tau) - \gamma^{-2}C_1^T(\tau)C_1(\tau))P(\tau), \quad P(\tau)|_{\tau=T} = 0. \end{aligned}$$

Taking adjoints to address the problem (23) leads to (22), for which the existence of a positive definite solution implies  $x^T(T)P(T)x(T) - x^T(0)P(0)x(0) +$

---

“Entrepreneurs must devote a portion of their minds to constantly processing uncertainty. So you sacrifice a degree of being present.” *Scott Branson*

$\int_0^T \tilde{y}^T(t|t) \tilde{y}(t|t) dt - \gamma^2 \int_0^T i^T(t) i(t) dt < 0$ . Thus, under the assumption  $x(0) = 0$ ,  
 $\int_0^T \tilde{y}^T(t|t) \tilde{y}(t|t) dt - \gamma^2 \int_0^T i^T(t) i(t) dt < -x^T(T) P(T) x(T) < 0$ . Therefore,  
 $\mathcal{R}_{yi} \in \mathcal{L}_\infty$ , that is,  $w, v \in \mathcal{L}_2 \Rightarrow \tilde{y} \in \mathcal{L}_2$ .

#### 9.2.4.4 Trading-Off $H_\infty$ Performance

In a robust filter design it is desired to meet an  $H_\infty$  performance objective for a minimum possible  $\gamma$ . A minimum  $\gamma$  can be found by conducting a search and checking for the existence of positive definite solutions to the Riccati differential equation (22). This search is tractable because  $\dot{P}(t)$  is a convex function of  $\gamma^2$ ,

$$\text{since } \frac{\partial^2 \dot{P}(t)}{\partial^2 \gamma^2} = \gamma^{-6} P(t) C_1^T(t) C_1(t) P(t) > 0.$$

In some applications it may be possible to estimate *a priori* values for  $\gamma$ . Recall for output estimation problems that the error is generated by  $\tilde{y} = \mathcal{R}_{yi} i$ , where  $\mathcal{R}_{yi} = -[\mathcal{H} \quad (\mathcal{H} - I)\mathcal{G}_1]$ . From the arguments of Chapters 1 – 2 and [28], for single-input-single-output plants  $\lim_{\sigma_v^2 \rightarrow 0} |\mathcal{H}| = 1$  and  $\lim_{\sigma_v^2 \rightarrow 0} |\mathcal{R}_{yi}| = 1$ , which implies

$$\lim_{\sigma_v^2 \rightarrow 0} \|\mathcal{R}_{yi} \mathcal{R}_{yi}^H\|_\infty = \sigma_v^2. \text{ Since the } H_\infty \text{ filter achieves the performance } \|\mathcal{R}_{yi} \mathcal{R}_{yi}^H\|_\infty <$$

$\gamma^2$ , it follows that an *a priori* design estimate is  $\gamma = \sigma_v$  at high signal-to-noise-ratios.

When the problem is stationary (or time-invariant), the filter gain is precalculated as  $K = PC^T R^{-1}$ , where  $P$  is the solution of the algebraic Riccati equation

$$0 = AP + PA^T - P(C_2^T R^{-1} C_2 - \gamma^{-2} C_1^T C_1)P + BQB^T. \quad (24)$$

Suppose that  $\mathcal{G}_2 = \mathcal{G}_1$  is a time-invariant single-input-single-output system and let  $R_{ei}(s)$  denote the transfer function of  $\mathcal{R}_{yi}$ . Then Parseval's Theorem states that the average total energy of  $e(t|t)$  is

$$\|\tilde{y}(t|t)\|_2^2 = \int_{-\infty}^{\infty} |\tilde{y}(t|t)|^2 dt = \int_{-\infty}^{\infty} R_{yi} R_{yi}^H(s) ds = \int_{-\infty}^{\infty} |R_{yi}(s)|^2 ds, \quad (25)$$

which equals the area under the error power spectral density,  $R_{yi} R_{yi}^H(s)$ . Recall that the optimal filter (in which  $\gamma = \infty$ ) minimises (25), whereas the  $H_\infty$  filter minimises

---

“Life is uncertain. Eat dessert first.” *Ernestine Ulmer*

$$\sup_{\|i\|_2 \neq 0} \|R_{\tilde{y}i} R_{\tilde{y}i}^H\|_{\infty} = \sup_{\|i\|_2 \neq 0} \frac{\|\tilde{y}\|_2^2}{\|i\|_2^2} < \gamma^2. \quad (26)$$

In view of (25) and (26), it follows that the  $H_{\infty}$  filter minimises the maximum magnitude of  $R_{\tilde{y}i} R_{\tilde{y}i}^H(s)$ . Consequently, it is also called a ‘minimax filter’. However, robust designs, which accommodate uncertain inputs tend to be conservative. Therefore, it is prudent to investigate using a larger  $\gamma$  to achieve a trade-off between  $H_{\infty}$  and minimum-mean-square-error performance criteria.

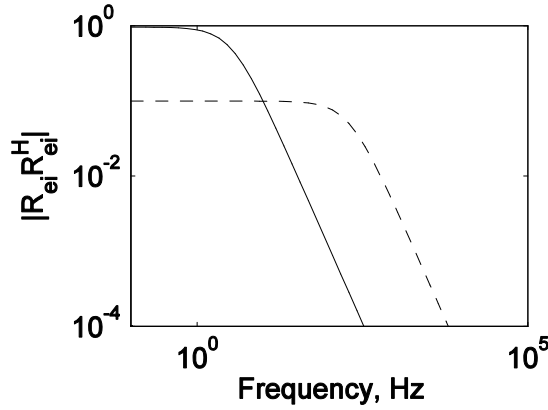


Fig. 2.  $|R_{\tilde{y}i} R_{\tilde{y}i}^H(s)|$  versus frequency for Example 1: optimal filter (solid line) and  $H_{\infty}$  filter (dotted line).

**Example 1.** Consider a time-invariant output estimation problem where  $A = -1$ ,  $B = C_2 = C_1 = 1$ ,  $\sigma_w^2 = 10$  and  $\sigma_v^2 = 0.1$ . The magnitude of the error spectrum exhibited by the optimal filter (designed with  $\gamma^2 = 10^8$ ) is indicated by the solid line of Fig. 2. From a search, a minimum of  $\gamma^2 = 0.099$  was found such that the algebraic Riccati equation (24) has a positive definite solution, which concurs with the *a priori* estimate of  $\gamma^2 \approx \sigma_v^2$ . The magnitude of the error spectrum exhibited by the  $H_{\infty}$  filter is indicated by the dotted line of Fig. 2. The figure demonstrates that the filter achieves  $|R_{ei} R_{ei}^H(s)| < \gamma^2$ . Although the  $H_{\infty}$  filter reduces the peak of the error spectrum by 10 dB, it can be seen that the area under the curve is larger, that is, the mean square error increases. Consequently, some intermediate value of  $\gamma$  may need to be considered to trade off peak error (spectrum) and average error performance.

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“If the uncertainty is larger than the effect, the effect itself becomes moot.” *Patrick Frank*

### 9.2.5 Accommodating Uncertainty

The above filters are designed for situations in which the inputs  $v(t)$  and  $w(t)$  are uncertain. Next, problems in which model uncertainty is present are discussed. The described approaches involve converting the uncertainty into a fictitious noise source and solving an auxiliary  $H_\infty$  filtering problem.

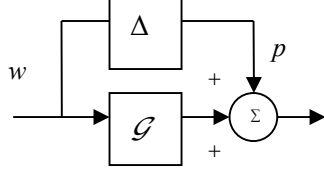


Fig. 3. Representation of additive model uncertainty.

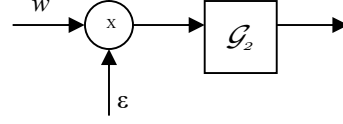


Fig. 4. Input scaling in lieu of a problem that possesses an uncertainty.

#### 9.2.5.1 Additive Uncertainty

Consider a time-invariant output estimation problem in which the nominal model is  $\mathcal{G}_2 + \Delta$ , where  $\mathcal{G}_2$  is known and  $\Delta$  is unknown, as depicted in Fig. 3. The  $p(t)$  represents a fictitious signal to account for discrepancies due to the uncertainty. It is argued below that a solution to the  $H_\infty$  filtering problem can be found by solving an auxiliary problem in which the input is scaled by  $\varepsilon \in \mathbb{R}$  as shown in Fig. 4. In lieu of the filtering problem possessing the uncertainty  $\Delta$ , an auxiliary problem is defined as

$$\dot{x}(t) = Ax(t) + Bw(t) + Bp(t), \quad x(0) = 0, \quad (27)$$

$$z(t) = C_2(t)x(t) + v(t), \quad (28)$$

$$\tilde{y}(t|t) = C_1(t)x(t) - C_1(t)\hat{x}(t), \quad (29)$$

where  $p(t)$  is an additional exogenous input satisfying

$$\|p\|_2^2 < \delta^2 \|w\|_2^2, \quad \delta \in \mathbb{R}. \quad (30)$$

Consider the scaled  $H_\infty$  filtering problem where

$$\dot{x}(t) = Ax(t) + B\varepsilon w(t), \quad x(0) = 0, \quad (31)$$

$$z(t) = C_2(t)x(t) + v(t), \quad (32)$$

$$\tilde{y}(t|t) = C_1(t)x(t) - C_1(t)\hat{x}(t), \quad (33)$$

---

“A theory has only the alternative of being right or wrong. A model has a third possibility - it may be right but irrelevant.” *Manfred Eigen*

in which  $\varepsilon^2 = (1 + \delta^2)^{-1}$ .

**Lemma 3 [26]:** Suppose for a  $\gamma \neq 0$  that the scaled  $H_\infty$  problem (31) – (33) is solvable, that is,  $\|\tilde{y}\|_2^2 < \gamma^2(\varepsilon^{-2}\|w\|_2^2 + \|v\|_2^2)$ . Then, this guarantees the performance

$$\|\tilde{y}\|_2^2 < \gamma^2(\|w\|_2^2 + \|p\|_2^2 + \|v\|_2^2) \quad (34)$$

for the solution of the auxiliary problem (27) – (29).

**Proof:** From the assumption that problem (31) – (33) is solvable, it follows that  $\|\tilde{y}\|_2^2 < \gamma^2(\varepsilon^{-2}\|w\|_2^2 + \|v\|_2^2)$ . Substituting for  $\varepsilon$ , using (30) and rearranging yields (34).

### 9.2.5.2 Multiplicative Uncertainty

Next, consider a filtering problem in which the model is  $G(I + \Delta)$ , as depicted in Fig. 5. It is again assumed that  $G$  and  $\Delta$  are known and unknown transfer function matrices, respectively. This problem may similarly be solved using Lemma 3. Thus a filter that accommodates additive or multiplicative uncertainty simply requires scaling of an input. The above scaling is only sufficient for a  $H_\infty$  performance criterion to be met. The design may well be too conservative and it is worthwhile to explore the merits of using values for  $\delta$  less than the uncertainty's assumed norm bound.

### 9.2.5.3 Parametric Uncertainty

Finally, consider a time-invariant output estimation problem in which the state matrix is uncertain, namely,

$$\dot{x}(t) = (A + \Delta_A)x(t) + Bw(t), \quad x(0) = 0, \quad (35)$$

$$z(t) = C_2(t)x(t) + v(t), \quad (36)$$

$$\tilde{y}(t | t) = C_1(t)x(t) - C_1(t)\hat{x}(t), \quad (37)$$

where  $\Delta_A \in \mathbb{R}^{n \times n}$  is unknown. Define an auxiliary  $H_\infty$  filtering problem by

$$\dot{x}(t) = Ax(t) + Bw(t) + p(t), \quad x(0) = 0, \quad (38)$$

(36) and (37), where  $p(t) = \Delta_A x(t)$  is a fictitious exogenous input. A solution to this problem would achieve

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“Remember that all models are wrong; the practical question is how wrong do they have to be to not be useful.” *George Edward Pelham Box*



$$\|\tilde{y}\|_2^2 < \gamma^2 (\|w\|_2^2 + \|p\|_2^2 + \|v\|_2^2) \quad (39)$$

for a  $\gamma \neq 0$ . From the approach of [14], [18], [19], consider the scaled filtering problem

$$\dot{x}(t) = Ax(t) + \bar{B}\bar{w}(t), \quad x(0) = 0, \quad (40)$$

(36), (37), where  $\bar{B} = \begin{bmatrix} B & \varepsilon^{-1} \end{bmatrix}$ ,  $\bar{w}(t) = \begin{bmatrix} w(t) \\ \varepsilon p(t) \end{bmatrix}$  and  $0 < \varepsilon < 1$ . Then the solution of this  $H_\infty$  filtering problem satisfies

$$\|\tilde{y}\|_2^2 < \gamma^2 (\|w\|_2^2 + \varepsilon^2 \|p\|_2^2 + \|v\|_2^2), \quad (41)$$

which implies (39). Thus, state matrix parameter uncertainty can be accommodated by including a scaled input in the solution of an auxiliary  $H_\infty$  filtering problem. Similar solutions to problems in which other state-space parameters are uncertain appear in [18], [19], [14].

## 9.2.6 Continuous-Time $H_\infty$ Smoothing

### 9.2.6.1 Background

There are three kinds of  $H_\infty$  smoothers: fixed point, fixed lag and fixed interval (see the tutorial [13]). The next development is concerned with continuous-time  $H_\infty$  fixed-interval smoothing. The smoother in [10] arises as a combination of forward states from an  $H_\infty$  filter and adjoint states that evolve according to a Hamiltonian matrix. A different fixed-interval smoothing problem to [10] is found in [16] by solving for saddle conditions within differential games. A summary of some filtering and smoothing results appears in [13]. Robust prediction, filtering and smoothing problems are addressed in [22]; the  $H_\infty$  predictor, filter and smoother require the solution of a Riccati differential equation that evolves forward in time, whereas the smoother additionally requires another to be solved in reverse-time. Another approach for combining forward and adjoint estimates is described [32] where the Fraser-Potter formula is used to construct a smoothed estimate.

Continuous-time, fixed-interval smoothers that differ from the formulations within [10], [13], [16], [22], [32] are reported in [34] – [35]. A robust version of [34] – [35] appears in [33], which is described below.

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“The purpose of models is not to fit the data but to sharpen the questions.” *Samuel Karlin*

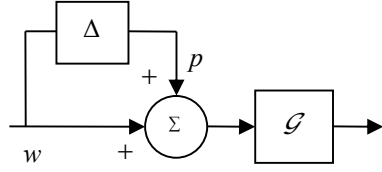


Fig. 5. Representation of multiplicative model uncertainty.

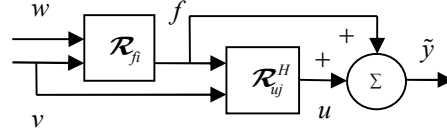


Fig. 6. Robust smoother error structure.

### 9.2.6.2 Problem Definition

Once again, it is assumed that the data is generated by (14) – (17). For convenience, attention is confined to output estimation, namely  $\mathcal{G}_2 = \mathcal{G}_1$  within Fig. 1. Input and state estimation problems can be handled similarly using the solution structures described in Chapter 6. It is desired to find a fixed-interval smoother solution  $\mathcal{H}$  that produces estimates  $\hat{y}_1(t|T)$  of  $y_1(t)$  so that the output estimation error

$$\tilde{y}(t|T) = y_1(t) - \hat{y}_1(t|T) \quad (42)$$

is in  $\mathcal{L}_2$ . As before, the map from the inputs  $i = \begin{bmatrix} v \\ w \end{bmatrix}$  to the error is denoted by

$\mathcal{R}_{yi} = -[\mathcal{H} \quad \mathcal{H}\mathcal{G}_1 - \mathcal{G}_1]$  and the objective is to achieve  $\int_0^T \tilde{y}^T(t|T)\tilde{y}(t|T) dt - \gamma^2 \int_0^T i^T(t)i(t) dt < 0$  for some  $\gamma \in \mathbb{R}$ .

### 9.2.6.3 $H_\infty$ Solution

The following  $H_\infty$  fixed-interval smoother exploits the structure of the minimum-variance smoother but uses the gain (21) calculated from the solution of the Riccati differential equation (22) akin to the  $H_\infty$  filter. An approximate Wiener-Hopf factor inverse,  $\hat{\Delta}^{-1}$ , is given by

$$\begin{bmatrix} \dot{\hat{x}}(t|t) \\ \alpha(t) \end{bmatrix} = \begin{bmatrix} A(t) - K(t)C(t) & K(t) \\ -R^{-1/2}(t)C(t) & R^{-1/2}(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t|t) \\ z(t) \end{bmatrix}. \quad (43)$$

An inspection reveals that the states within (43) are the same as those calculated by the  $H_\infty$  filter (19). The adjoint of  $\hat{\Delta}^{-1}$ , which is denoted by  $\hat{\Delta}^{-H}$ , has the realisation

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“Certainty is the mother of quiet and repose, and uncertainty the cause of variance and contentions.”  
Edward Coke

$$\begin{bmatrix} -\dot{\xi}(t) \\ \beta(t) \end{bmatrix} = \begin{bmatrix} A^T(t) - C^T(t)K^T(t) & -C^T(t)R^{-1/2}(t) \\ K^T(t) & R^{-1/2}(t) \end{bmatrix} \begin{bmatrix} \xi(t) \\ \alpha(t) \end{bmatrix}. \quad (44)$$

Output estimates are obtained as

$$\hat{y}(t|T) = z(t) - R(t)\beta(t). \quad (45)$$

However, an additional condition requires checking in order to guarantee that the smoother actually achieves the above performance objective; the existence of a solution  $P_2(t) = P_2^T(t) > 0$  is required for the auxiliary Riccati differential equation

$$\begin{aligned} -\dot{P}_2(t) &= \bar{A}(t)P_2(t) + P_2(t)\bar{A}^T(t) + K(t)R^2(t)K^T(t) \\ &\quad + \gamma_2^{-2}P_2(t)C^T(t)R^{-1}(t)(C(t)P(t)C^T(t) + R(t))R^{-1}(t)C(t)P_2(t), \\ P_2(T) &= 0, \end{aligned} \quad (46)$$

where  $\bar{A}(t) = A(t) - K(t)C(t)$ .

### 9.2.7 Performance

It will be shown subsequently that the robust fixed-interval smoother (43) – (45) has the error structure shown in Fig. 6, which is examined below.

**Lemma 4 [33]:** Consider the arrangement of two linear systems  $f = \mathcal{R}_{ji}i$  and  $u =$

$\mathcal{R}_{ij}^H j$  shown in Fig. 6, in which  $i = \begin{bmatrix} w \\ v \end{bmatrix}$  and  $j = \begin{bmatrix} f \\ v \end{bmatrix}$ . Let  $\mathcal{R}_{yi}$  denote the map from  $i$  to  $\tilde{y}$ . Assume that  $w$  and  $v \in \mathcal{L}_2$ . If and only if: (i)  $\mathcal{R}_{ji} \in \mathcal{L}_\infty$  and (ii)  $\mathcal{R}_{ij}^H \in \mathcal{L}_\infty$ , then (i)  $f, u, \tilde{y} \in \mathcal{L}_2$  and (ii)  $\mathcal{R}_{yi} \in \mathcal{L}_\infty$ .

**Proof:** (i) To establish sufficiency, note that  $\|i\|_2 \leq \|w\|_2 + \|v\|_2 \Rightarrow i \in \mathcal{L}_2$ , which with Condition (i)  $\Rightarrow f \in \mathcal{L}_2$ . Similarly,  $\|j\|_2 \leq \|f\|_2 + \|v\|_2 \Rightarrow j \in \mathcal{L}_2$ , which with Condition (ii)  $\Rightarrow u \in \mathcal{L}_2$ . Also,  $\|\tilde{y}\|_2 \leq \|f\|_2 + \|u\|_2 \Rightarrow \tilde{y} \in \mathcal{L}_2$ . The necessity of (i) follows from the assumption  $i \in \mathcal{L}_2$  together with the property

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“For, Mathematical Demonstrations being built upon the impregnable Foundations of Geometry and Arithmetick, are the only Truths, that can sink into the Mind of Man, void of all Uncertainty; and all other Discourses participate more or less of Truth, according as their Subjects are more or less capable of Mathematical Demonstration.” *Sir Christopher Wen*

$\mathcal{R}_{f_i} \mathcal{L}_2 \subset \mathcal{L}_2 \Rightarrow \mathcal{R}_{f_i} \in \mathcal{L}_\infty$  (see [p. 83, 21]). Similarly,  $j \in \mathcal{L}_2$  together with the property  $\mathcal{R}_{ij}^H \mathcal{L}_2 \subset \mathcal{L}_2 \Rightarrow \mathcal{R}_{ij}^H \in \mathcal{L}_\infty$ .

(ii) Finally,  $i \in \mathcal{L}_2$ ,  $\tilde{y} = \mathcal{R}_{yi} i \in \mathcal{L}_2$  together with the property  $\mathcal{R}_{yi} \mathcal{L}_2 \subset \mathcal{L}_2 \Rightarrow \mathcal{R}_{yi} \in \mathcal{L}_\infty$ .

It is easily shown that the error system,  $\mathcal{R}_{ei}$ , for the model (14) – (15), the data (17) and the smoother (43) – (45), is given by

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ -\dot{\xi}(t) \\ \tilde{y}(t|T) \end{bmatrix} = \begin{bmatrix} \bar{A}(t) & 0 & B(t) & -K(t) \\ -C^T(t)R^{-1}(t)C(t) & \bar{A}^T(t) & 0 & -C^T(t)R^{-1}(t) \\ C(t) & R(t)K^T(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \xi(t) \\ w(t) \\ v(t) \end{bmatrix}, \quad (47)$$

$$\tilde{x}(0) = 0, \quad \xi(T) = 0,$$

where  $\tilde{x}(t|t) = x(t) - \hat{x}(t|t)$ . The conditions for the smoother attaining the desired performance<sup>15</sup> objective are described below.

**Lemma 5 [33]:** *In respect of the smoother error system (47), if there exist symmetric positive definite solutions to (22) and (46) for  $\gamma, \gamma_2 > 0$ , then the smoother (43) – (45) achieves  $\mathcal{R}_{yi} \in \mathcal{L}_\infty$ , that is,  $i \in \mathcal{L}_2$  implies  $\tilde{y} \in \mathcal{L}_2$ .*

**Proof:** Since  $\tilde{x}(t|t)$  is decoupled from  $\xi(t)$ ,  $\mathcal{R}_{yi}$  is equivalent to the arrangement of two systems  $\mathcal{R}_{fi}$  and  $\mathcal{R}_{ij}^H$  shown in Fig. 6. The  $\mathcal{R}_{fi}$  is defined by (23) in which  $C_2(t) = C(t)$ . From Lemma 2, the existence of a positive definite solution to (22) implies  $\mathcal{R}_{fi} \in \mathcal{L}_\infty$ . The  $\mathcal{R}_{ij}^H$  is given by the system

$$-\dot{\xi}(\tau) = \bar{A}^T(\tau)\xi(\tau) - C^T(\tau)R^{-1}(\tau)\tilde{y}(\tau|\tau) - C^T(\tau)R^{-1}(\tau)v(\tau), \quad \xi(T) = 0, \quad (48)$$

$$u(\tau) = R(\tau)K^T(\tau)\xi(\tau). \quad (49)$$

For the above system to be in  $\mathcal{L}_\infty$ , from Lemma 4, it is required that there exists a solution to (46) for which the existence of a positive definite solution implies  $\mathcal{R}_{ij}^H \in \mathcal{L}_\infty$ . The claim  $\mathcal{R}_{yi} \in \mathcal{L}_\infty$  follows from Lemma 4.

The  $H_\infty$  solution can be derived as a solution to a two-point boundary value problem, which involves a trade-off between causal and noncausal processes (see [10], [15], [21]). This suggests that the  $H_\infty$  performance of the above smoother would not improve on that of the filter. Indeed, from Fig. 6,  $\tilde{y} = f + u$  and the

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<sup>15</sup>“Doubt is uncomfortable, certainty is ridiculous.” *François-Marie Arouet de Voltaire*

triangle rule yields  $\|\tilde{y}\|_2 \leq \|f\|_2 + \|u\|_2$ , where  $f$  is the  $H_\infty$  filter error. That is, the error upper bound for the  $H_\infty$  fixed-interval smoother (43) – (45) is greater than that for the  $H_\infty$  filter (19) – (20). It is observed below that compared to the minimum-variance case, the  $H_\infty$  solution exhibits an increased mean-square error.

**Lemma 6 [33]:** For the output estimation problem (14) – (18), in which  $C_2(t) = C_1(t) = C(t)$ , the smoother solution (43) – (45) results in

$$\|\mathcal{R}_{\tilde{y}i} \mathcal{R}_{\tilde{y}i}^H\|_{2, \gamma^{-2} > 0} > \|\mathcal{R}_{\tilde{y}i} \mathcal{R}_{\tilde{y}i}^H\|_{2, \gamma^{-2} = 0}. \quad (50)$$

**Proof:** By expanding  $\mathcal{R}_{\tilde{y}i} \mathcal{R}_{\tilde{y}i}^H$  and completing the squares, it can be shown that  $\mathcal{R}_{\tilde{y}i} \mathcal{R}_{\tilde{y}i}^H = \mathcal{R}_{\tilde{y}i1} \mathcal{R}_{\tilde{y}i1}^H + \mathcal{R}_{\tilde{y}i2} \mathcal{R}_{\tilde{y}i2}^H$ , where  $\mathcal{R}_{\tilde{y}i2} \mathcal{R}_{\tilde{y}i2}^H = \mathcal{G}_1 Q(t) \mathcal{G}_1^H - \mathcal{G}_1 Q(t) \mathcal{G}_1^H \Delta^{-H} \Delta^{-1} \mathcal{G}_1 Q(t) \mathcal{G}_1^H$  and  $\mathcal{R}_{\tilde{y}i1} = \mathcal{H} \Delta - \mathcal{G}_1 Q(t) \mathcal{G}_1^H \Delta^{-H} = [\mathcal{H} - I + R(t)(\Delta \Delta^H)^{-1}] \Delta$ . Substituting  $\mathcal{H} = I - R(t)(\hat{\Delta} \hat{\Delta}^H)^{-1}$  into  $\mathcal{R}_{\tilde{y}i1}$  yields

$$\mathcal{R}_{\tilde{y}i1} = R(t)[(\Delta \Delta^H)^{-1} - (\hat{\Delta} \hat{\Delta}^H)^{-1}] \Delta, \quad (51)$$

which suggests  $\hat{\Delta} = C(t) \mathcal{G}_0 K(t) R^{1/2}(t) + R^{1/2}(t)$ , where  $\mathcal{G}_0$  denotes an operator having the state-space realisation  $\begin{bmatrix} A(t) & I \\ I & 0 \end{bmatrix}$ . Constructing  $\hat{\Delta} \hat{\Delta}^H = C(t) \mathcal{G}_0 [K(t) R(t) K^T(t) - P(t) A^T(t) - A(t) P(t)] \mathcal{G}_0^H C^T(t) + R(t)$  and using (22) yields  $\hat{\Delta} \hat{\Delta}^H = C(t) \mathcal{G}_0 [B(t) Q(t) B^T(t) - \dot{P}(t) + \gamma^{-2} P(t) C^T(t) C(t) P(t)] \mathcal{G}_0^H C^T(t) + R(t)$ . Comparison with  $\Delta \Delta^H = C(t) \mathcal{G}_0 B(t) Q(t) B^T(t) \mathcal{G}_0^H C^T(t) + R(t)$  leads to  $\hat{\Delta} \hat{\Delta}^H = \Delta \Delta^H - C(t) \mathcal{G}_0 (\dot{P}(t) + \gamma^{-2} P(t) C^T(t) C(t) P(t)) \mathcal{G}_0^H C^T(t)$ . Substituting for  $\hat{\Delta} \hat{\Delta}^H$  into (51) yields

$$\begin{aligned} \mathcal{R}_{\tilde{y}i1} &= R(t)[(\Delta \Delta^H)^{-1} - (\Delta \Delta^H - C(t) \mathcal{G}_0 (\dot{P}(t) \\ &\quad - \gamma^{-2} P(t) C^T(t) C(t) P(t)) \mathcal{G}_0^H C^T(t))^{-1}] \Delta. \end{aligned} \quad (52)$$

The observation (50) follows by inspection of (52).

Thus, the cost of designing for worst case input conditions is a deterioration in the mean performance. Note that the best possible average performance  $\|\mathcal{R}_{\tilde{y}i} \mathcal{R}_{\tilde{y}i}^H\|_2 = \|\mathcal{R}_{\tilde{y}i2} \mathcal{R}_{\tilde{y}i2}^H\|_2$  can be attained in problems where there are no uncertainties

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“I am not accustomed to saying anything with certainty after only one or two observations.” *Andreas Vesalius*

present,  $\gamma^{-2} = 0$  and the Riccati equation solution has converged, that is,  $\dot{P}(t) = 0$ , in which case  $\hat{\Delta}\hat{\Delta}^H = \Delta\Delta^H$  and  $\mathcal{R}_{y|l}$  is a zero matrix.

### 9.2.8 Performance Comparison

It is of interest to compare to compare the performance of (43) – (45) with the  $H_\infty$  smoother described in [10], [13], [16], namely,

$$\begin{bmatrix} \dot{\hat{x}}(t|T) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t)Q(t)B^T(t) \\ C^T(t)R^{-1}(t)C(t) & -A'(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t|T) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -C^T(t)R^{-1}(t) \end{bmatrix} z(t), \quad (53)$$

$$\hat{x}(t|T) = \hat{x}(t) + P(t)\xi(t) \quad (54)$$

and (22). Substituting (54) and its differential into the first row of (53) together with (21) yields

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)(z(t) - C(t)\hat{x}(t)), \quad (55)$$

which reverts to the Kalman filter at  $\gamma^{-2} = 0$ . Substituting  $\xi(t) = P^{-1}(t)(\hat{x}(t|T) - \hat{x}(t))$  into the second row of (53) yields

$$\dot{\hat{x}}(t|T) = A(t)\hat{x}(t) + G(t)(\hat{x}(t|T) - \hat{x}(t)), \quad (56)$$

where  $G(t) = B(t)Q(t)B^T(t)P_2^{-1}(t)$ , which reverts to the maximum-likelihood smoother at  $\gamma^{-2} = 0$ . Thus, the Hamiltonian form (53) – (54) can be realised by calculating the filtered estimate (55) and then obtaining the smoothed estimate from (56).

**Example: 2 [35].** Let  $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $B = C = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  and  $R =$

$\begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$  denote time-invariant parameters for an output estimation problem.

Simulations were conducted for the case of  $T = 100$  seconds,  $\delta t = 1$  millisecond, using 500 realisations of zero-mean, Gaussian process noise and measurement noise. The resulting mean-square-error (MSE) versus signal-to-noise ratio (SNR) are shown in Fig. 7. The  $H_\infty$  solutions were calculated using *a priori* designs of

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“We know accurately only when we know little, with knowledge doubt increases.” *Johann Wolfgang von Goethe*

$\gamma^{-2} = \sigma_v^{-2}$  within (22). It can be seen from trace (vi) of Fig. 7 that the  $H_\infty$  smoothers exhibit poor performance when the exogenous inputs are in fact Gaussian, which illustrates Lemma 6. The figure demonstrates that the minimum-variance smoother out-performs the maximum-likelihood smoother. However, at high SNR, the difference in smoother performance is inconsequential. Intermediate values for  $\gamma^{-2}$  may be selected to realise a smoother design that achieves a trade-off between minimum-variance performance (trace (iii)) and  $H_\infty$  performance (trace (v)).

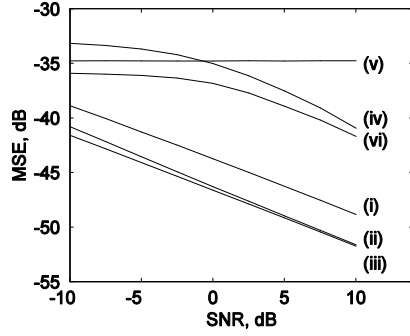


Fig. 7. Fixed-interval smoother performance comparison for Gaussian process noise: (i) Kalman filter; (ii) Maximum likelihood smoother; (iii) Minimum-variance smoother; (iv)  $H_\infty$  filter; (v)  $H_\infty$  smoother [10], [13], [16]; and (vi)  $H_\infty$  smoother (43) – (45).

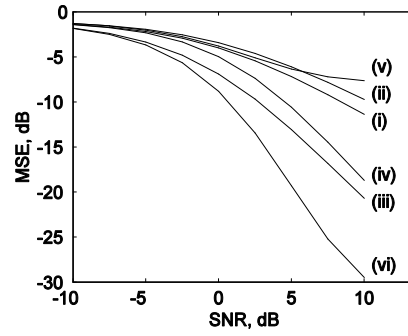


Fig. 8. Fixed-interval smoother performance comparison for sinusoidal process noise: (i) Kalman filter; (ii) Maximum likelihood smoother; (iii) Minimum-variance smoother; (iv)  $H_\infty$  filter; (v)  $H_\infty$  smoother [10], [13], [16]; and (vi)  $H_\infty$  smoother (43) – (45).

**Example 3 [35].** As in Example 6.6, consider the non-Gaussian process noise signal  $w(t) = \sin(t)\sigma_{\sin(t)}^{-1}$ , where  $\sigma_{\sin(t)}^2$  denotes the sample variance of  $\sin(t)$ . The results of a simulation study appear in Fig. 8. It can be seen that the  $H_\infty$  solutions, which accommodate input uncertainty, perform better than those relying on Gaussian noise assumptions. In this example, the developed  $H_\infty$  smoother (43) – (45) exhibits the best mean-square-error performance.

## 9.3 Robust Discrete-Time Estimation

### 9.3.3 Discrete-Time Bounded Real Lemma

The development of discrete-time  $H_\infty$  filters and smoothers proceeds analogously to the continuous-time case. From Lyapunov stability theory [36], for the unforced system

$$x_{k+1} = A_k x_k, \quad (57)$$

$A_k \in \mathbb{R}^{n \times n}$ , to be asymptotically stable over the interval  $k \in [1, N]$ , a Lyapunov function,  $V_k(x_k)$ , is required to satisfy  $\Delta V_k(x_k) < 0$ , where  $\Delta V_k(x_k) = V_{k+1}(x_k) - V_k(x_k)$  denotes the first backward difference of  $V_k(x_k)$ . Consider the candidate Lyapunov function  $V_k(x_k) = x_k^T P_k x_k$ , where  $P_k = P_k^T \in \mathbb{R}^{n \times n}$  is positive definite. To guarantee  $x_k \in \ell_2$ , it is required that

$$\Delta V_k(x_k) = x_{k+1}^T P_{k+1} x_{k+1} - x_k^T P_k x_k < 0. \quad (58)$$

Now let  $y = \mathcal{G} w$  denote the output of the system

$$x_{k+1} = A_k x_k + B_k w_k, \quad (59)$$

$$y_k = C_k x_k, \quad (60)$$

where  $w_k \in \mathbb{R}^m$ ,  $B_k \in \mathbb{R}^{n \times m}$  and  $C_k \in \mathbb{R}^{p \times n}$ .

The Bounded Real Lemma [18] states that  $w \in \ell_2$  implies  $y \in \ell_2$  if

$$x_{k+1}^T P_{k+1} x_{k+1} - x_k^T P_k x_k + y_k^T y_k - \gamma^2 w_k^T w_k < 0 \quad (61)$$

for a  $\gamma \in \mathbb{R}$ . Summing (61) from  $k = 0$  to  $k = N - 1$  yields the objective

$$-x_0^T P_0 x_0 + \sum_{k=0}^{N-1} y_k^T y_k - \gamma^2 \sum_{k=0}^{N-1} w_k^T w_k < 0, \quad (62)$$

that is,

$$\frac{-x_0^T P_0 x_0 + \sum_{k=0}^{N-1} y_k^T y_k}{\sum_{k=0}^{N-1} w_k^T w_k} < \gamma^2. \quad (63)$$

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“Education is the path from cocky ignorance to miserable uncertainty.” *Samuel Langhorne Clemens aka. Mark Twain*



Assuming that  $x_0 = 0$ ,

$$\|\mathcal{G}\|_\infty = \frac{\|y\|_2}{\|w\|_2} = \frac{\|\mathcal{G}w\|_2}{\|w\|_2} = \frac{\sqrt{\sum_{k=0}^{N-1} y_k^T y_k}}{\sqrt{\sum_{k=0}^{N-1} w_k^T w_k}} < \gamma. \quad (64)$$

Conditions for achieving the above objectives are established below.

**Lemma 7:** *The discrete-time Bounded Real Lemma [18]: In respect of the above system  $\mathcal{G}$ , suppose that the Riccati difference equation*

$$P_k = A_k^T P_{k+1} A_k + \gamma^{-2} A_k^T P_{k+1} B_k (I - \gamma^{-2} B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k + C_k^T C_k, \quad (65)$$

with  $P_T = 0$ , has a positive definite symmetric solution on  $[0, N]$ . Then  $\|\mathcal{G}\|_\infty \leq \gamma$  for any  $w \in \ell_2$ .

**Proof:** From the approach of Xie et al [18], define

$$p_k = w_k - \gamma^{-2} (I - \gamma^{-2} B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k x_k. \quad (66)$$

It is easily verified that

$$\begin{aligned} & x_{k+1}^T P_{k+1} x_{k+1} - x_k^T P_k x_k + y_k^T y_k - \gamma^2 w_k^T w_k \\ &= -\gamma^{-2} p_k^T (I - \gamma^{-2} B_k^T P_{k+1} B_k)^{-1} p_k - x_k^T A_k^T P_{k+1} A_k x_k, \end{aligned}$$

which implies (61) – (62) and (63) under the assumption  $x_0 = 0$ .

The above lemma relies on the simplifying assumption  $E\{w_j w_k^T\} = I \delta_{jk}$ . When  $E\{w_j w_k^T\} = Q_k \delta_{jk}$ , the scaled matrix  $\bar{B}_k = B_k Q_k^{1/2}$  may be used in place of  $B_k$  above. In the case where  $\mathcal{G}$  possesses a direct feedthrough matrix, namely,  $y_k = C_k x_k + D_k w_k$ , the Riccati difference equation within the above lemma becomes

$$\begin{aligned} P_k &= A_k^T P_{k+1} A_k + C_k^T C_k \\ &+ \gamma^{-2} (A_k^T P_{k+1} B_k + C_k^T D_k) (I - \gamma^{-2} B_k^T P_{k+1} B_k - \gamma^{-2} D_k^T D_k)^{-1} (B_k^T P_{k+1} A_k + D_k^T C_k). \end{aligned} \quad (67)$$

A verification is requested in the problems. It will be shown that predictors, filters and smoothers satisfy a  $H_\infty$  performance objective if there exist solutions to Riccati difference equations arising from the application of Lemma 7 to the

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“And as he thus spake for himself, Festus said with a loud voice, Paul, thou art beside thyself; much learning doth make thee mad.” *Acts 26: 24*

corresponding error systems. A summary of the discrete-time results from [5], [11], [13] and the further details described in [21], [30], is presented below.

### 9.3.4 Discrete-Time $H_\infty$ Prediction

#### 9.3.4.1 Problem Definition

Consider a nominal system  $\mathcal{G}_2$

$$x_{k+1} = A_k x_k + B_k w_k, \quad (68)$$

$$y_{2,k} = C_{2,k} x_k, \quad (69)$$

together with a fictitious reference system  $\mathcal{G}_1$  realised by (68) and

$$y_{1,k} = C_{1,k} x_k, \quad (70)$$

where  $A_k$ ,  $B_k$ ,  $C_{2,k}$  and  $C_{1,k}$  are of appropriate dimensions. The problem of interest is to find a solution  $\mathcal{H}$  that produces one-step-ahead predictions,  $\hat{y}_{1,k/k-1}$ , given measurements

$$z_k = y_{2,k} + v_k \quad (71)$$

at time  $k - 1$ . The prediction error is defined as

$$\tilde{y}_{k/k-1} = y_{1,k} - \hat{y}_{1,k/k-1}. \quad (72)$$

The error sequence (72) is generated by  $\tilde{y} = \mathcal{R}_{\tilde{y}i}$ , where  $\mathcal{R}_{\tilde{y}i} = -[\mathcal{H} \quad \mathcal{H}\mathcal{G}_2 - \mathcal{G}_1]$ ,  $i = \begin{bmatrix} v \\ w \end{bmatrix}$  and the objective is to achieve  $\sum_{k=0}^{N-1} \tilde{y}_{k/k-1}^T \tilde{y}_{k/k-1} - \gamma^2 \sum_{k=0}^{N-1} i_k^T i_k < 0$ , for some  $\gamma \in \mathbb{R}$ . For convenience, it is assumed that  $w_k \in \mathbb{R}^m$ ,  $E\{w_k\} = 0$ ,  $E\{w_j w_k^T\} = Q_k \delta_{jk}$ ,  $v_k \in \mathbb{R}^p$ ,  $E\{v_k\} = 0$ ,  $E\{v_j v_k^T\} = R_k \delta_{jk}$  and  $E\{w_j v_k^T\} = 0$ .

#### 9.3.4.2 $H_\infty$ Solution

The  $H_\infty$  predictor has the same structure as the optimum minimum-variance (or Kalman) predictor. It is given by

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“Why waste time learning when ignorance is instantaneous?” *William Boyd Watterson II*

$$\hat{x}_{k+1/k} = (A_k - K_k C_{2,k}) \hat{x}_{k/k-1} + K_k z_k, \quad (73)$$

$$\hat{y}_{1,k/k-1} = C_{1,k} \hat{x}_{k/k-1}, \quad (74)$$

where

$$K_k = A_k P_{k/k-1} C_{2,k}^T (C_{2,k} P_{k/k-1} C_{2,k}^T + R_k)^{-1} \quad (75)$$

is the one-step-ahead predictor gain,

$$P_{k/k-1} = (M_k^{-1} - \gamma^{-2} C_{1,k+1}^T C_{1,k+1})^{-1}, \quad (76)$$

and  $M_k = M_k^T > 0$  satisfies the Riccati differential equation

$$\begin{aligned} M_{k+1} = & A_k M_k A_k^T + B_k Q_k B_k^T \\ & - A_k M_k \begin{bmatrix} C_{1,k}^T & C_{2,k}^T \end{bmatrix} \begin{bmatrix} C_{1,k} M_k C_{1,k}^T - \gamma^2 I & C_{1,k} M_k C_{2,k}^T \\ C_{2,k} M_k C_{1,k}^T & R_k + C_{2,k} M_k C_{2,k}^T \end{bmatrix}^{-1} \begin{bmatrix} C_{1,k} \\ C_{2,k} \end{bmatrix} M_k A_k^T \end{aligned} \quad (77)$$

such that  $\begin{bmatrix} C_{1,k} M_k C_{1,k}^T - \gamma^2 I & C_{1,k} M_k C_{2,k}^T \\ C_{2,k} M_k C_{1,k}^T & R_k + C_{2,k} M_k C_{2,k}^T \end{bmatrix} > 0$ . The above predictor is also known as an *a priori* filter within [11], [13], [30].

### 9.3.4.3 Performance

Following the approach in the continuous-time case, by subtracting (73) – (74) from (68), (70), the predictor error system is

$$\begin{aligned} \begin{bmatrix} \tilde{x}_{k+1/k} \\ \tilde{y}_{k/k-1} \end{bmatrix} &= \begin{bmatrix} A_k - K_k C_{2,k} & [-K_k & B_k] \\ C_{1,k} & [0 & 0] \end{bmatrix} \begin{bmatrix} \tilde{x}_{k/k-1} \\ v_k \\ w_k \end{bmatrix}, \quad \tilde{x}_0 = 0, \\ &= \mathcal{R}_{yi} i, \end{aligned} \quad (78)$$

where  $\tilde{x}_{k/k-1} = x_k - \hat{x}_{k/k-1}$ ,  $\mathcal{R}_{ei} = \begin{bmatrix} A_k - K_k C_{2,k} & [-K_k & B_k] \\ C_{1,k} & [0 & 0] \end{bmatrix}$  and  $i = \begin{bmatrix} v \\ w \end{bmatrix}$ . It is shown below that the prediction error satisfies the desired performance objective.

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“Give me a fruitful error any time, full of seeds bursting with its own corrections. You can keep your sterile truth for yourself.” *Vilfredo Federico Damaso Pareto*

**Lemma 8 [11], [13], [30]:** In respect of the  $H_\infty$  prediction problem (68) – (72), the existence of  $M_k = M_k^T > 0$  for the Riccati differential equation (77) ensures that the solution (73) – (74) achieves the performance objective  $\sum_{k=0}^{N-1} \tilde{y}_{k/k-1}^T \tilde{y}_{k/k-1} -$

$$\gamma^2 \sum_{k=0}^{N-1} i_k^T i_k < 0.$$

**Proof:** By applying the Bounded Real Lemma to  $\mathcal{R}_{yi}^H$  and taking the adjoint to address  $\mathcal{R}_{yi}$ , it is required that there exists a positive definite symmetric solution to

$$\begin{aligned} P_{k+1} &= (A_k - K_k C_{2,k}) P_k (A_k - K_k C_{2,k})^T + K_k R_k K_k^T + B_k Q_k B_k^T \\ &\quad + \gamma^{-2} (A_k - K_k C_{2,k}) P_k C_{1,k}^T (I - \gamma^{-2} C_{1,k} P_k C_{1,k}^T)^{-1} C_{2,k} P_k (A_k - K_k C_{2,k})^T \\ &= (A_k - K_k C_{2,k}) (P_k + \gamma^{-2} P_k C_{1,k}^T (I - \gamma^{-2} C_{1,k} P_k C_{1,k}^T)^{-1} C_{1,k} P_k) (A_k - K_k C_{2,k})^T \\ &\quad + K_k R_k K_k^T + B_k Q_k B_k^T \\ &= (A_k - K_k C_{2,k}) (P_k^{-1} + \gamma^{-2} C_{1,k}^T C_{1,k})^{-1} (A_k - K_k C_{2,k})^T + K_k R_k K_k^T + B_k Q_k B_k^T \end{aligned} \quad (79)$$

in which use was made of the Matrix Inversion Lemma. Defining  $P_{k/k-1} = (P_k^{-1} + \gamma^{-2} C_{1,k}^T C_{1,k})^{-1}$  leads to

$$\begin{aligned} (P_{k+1/k}^{-1} + \gamma^{-2} C_{1,k+1}^T C_{1,k+1})^{-1} &= (A_k - K_k C_{2,k}) P_{k/k-1} (A_k - K_k C_{2,k})^T + K_k R_k K_k^T + B_k Q_k B_k^T \\ &= A_k P_{k/k-1} A_k^T + B_k Q_k B_k^T - A_k P_{k/k-1} C_{2,k}^T (R + C_{2,k} P_{k/k-1} C_{2,k}^T)^{-1} C_{2,k} P_{k/k-1} A_k^T. \end{aligned}$$

and applying the Matrix Inversion Lemma gives

$$\begin{aligned} (P_{k+1/k}^{-1} + \gamma^{-2} C_{1,k+1}^T C_{1,k+1})^{-1} &= A_k P_{k/k-1} A_k^T + B_k Q_k B_k^T - A_k P_{k/k-1} C_{2,k}^T (R + C_{2,k} P_{k/k-1} C_{2,k}^T)^{-1} C_{2,k} P_{k/k-1} A_k^T \\ &= A_k (P_{k/k-1} + C_{2,k}^T R_k^{-1} C_{2,k})^{-1} A_k^T + B_k Q_k B_k^T. \end{aligned}$$

The change of variable (76), namely,  $P_{k/k-1}^{-1} = M_k^{-1} - \gamma^{-2} C_{1,k}^T C_{1,k}$ , results in

$$\begin{aligned} M_{k+1} &= A_k (M_k^{-1} + C_{2,k}^T R_k^{-1} C_{2,k} - \gamma^{-2} C_{1,k}^T C_{1,k})^{-1} A_k^T + B_k Q_k B_k^T \\ &= A_k (M_k^{-1} + \bar{C}_k \bar{R}_k \bar{C}_k^T)^{-1} A_k^T + B_k Q_k B_k^T. \end{aligned} \quad (80)$$

where  $\bar{C}_k = \begin{bmatrix} C_{1,k} \\ C_{2,k} \end{bmatrix}$  and  $\bar{R}_k = \begin{bmatrix} -\gamma^{-2}I & 0 \\ 0 & R_k \end{bmatrix}$ . Applying the Matrix Inversion Lemma within (80) gives

$$M_{k+1} = A_k M_k^T - A_k M_k \bar{C}_k^T (\bar{R}_k + \bar{C}_k M_k \bar{C}_k^T)^{-1} \bar{C}_k M_k A_k^T + B_k Q_k B_k^T, \quad (81)$$

Expanding (81) yields (77). The existence of  $M_k > 0$  for the above Riccati differential equation implies  $P_k > 0$  for (79). Thus, it follows from Lemma 7 that the stated performance objective is achieved.

### 9.3.5 Discrete-Time $H_\infty$ Filtering

#### 9.3.5.1 Problem Definition

Consider again the configuration of Fig. 1. Assume that the systems  $\mathcal{G}_2$  and  $\mathcal{G}_1$  have the realisations (68) – (69) and (68), (70), respectively. It is desired to find a solution  $\mathcal{H}$  that operates on the measurements (71) and produces the filtered estimates  $\hat{y}_{1,k/k}$ . The filtered error sequence,

$$\tilde{y}_{k/k} = y_{1,k} - \hat{y}_{1,k/k}, \quad (82)$$

is generated by  $\tilde{y} = \mathcal{R}_{\tilde{y}} i$ , where  $\mathcal{R}_{\tilde{y}} = -[\mathcal{H} \quad \mathcal{H}\mathcal{G}_2 - \mathcal{G}_1]$ ,  $i = \begin{bmatrix} v \\ w \end{bmatrix}$ . The  $H_\infty$

performance objective is to achieve  $\sum_{k=0}^{N-1} \tilde{y}_{k/k}^T \tilde{y}_{k/k} - \gamma^2 \sum_{k=0}^{N-1} i_k^T i_k < 0$ , for some  $\gamma \in \mathbb{R}$ .

#### 9.3.5.2 $H_\infty$ Solution

As explained in Chapter 4, filtered states can be evolved from

$$\hat{x}_{k/k} = A_{k-1} \hat{x}_{k-1/k-1} + L_k (z_k - C_{2,k} A_{k-1} \hat{x}_{k-1/k-1}), \quad (83)$$

where  $L_k \in \mathbb{R}^{n \times p}$  is a filter gain. The above recursion is called an *a posteriori* filter in [11], [13], [30]. Output estimates are obtained from

$$\hat{y}_{1,k-1/k-1} = C_{1,k-1} \hat{x}_{k-1/k-1}. \quad (84)$$

The filter gain is calculated as

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“I believe the most solemn duty of the American president is to protect the American people. If America shows uncertainty and weakness in this decade, the world will drift toward tragedy. This will not happen on my watch.” *George Walker Bush*

$$L_k = M_k C_k^T (C_k M_k C_k^T + R_k)^{-1}, \quad (85)$$

where  $M_k = M_k^T > 0$  satisfies the Riccati differential equation

$$\begin{aligned} M_k &= A_{k-1} M_{k-1} A_{k-1}^T + B_{k-1} Q_{k-1} B_{k-1}^T - A_{k-1} M_{k-1} \begin{bmatrix} C_{1,k-1}^T & C_{2,k-1}^T \end{bmatrix} \\ &\times \begin{bmatrix} C_{1,k-1} M_{k-1} C_{1,k-1}^T - \gamma^2 I & C_{1,k-1} M_{k-1} C_{2,k-1}^T \\ C_{2,k-1} M_{k-1} C_{1,k-1}^T & R_{k-1} + C_{2,k-1} M_{k-1} C_{2,k-1}^T \end{bmatrix}^{-1} \begin{bmatrix} C_{1,k-1} \\ C_{2,k-1} \end{bmatrix} M_{k-1} A_{k-1}^T \quad (86) \end{aligned}$$

such that  $\begin{bmatrix} C_{1,k-1} M_{k-1} C_{1,k-1}^T - \gamma^2 I & C_{1,k-1} M_{k-1} C_{2,k-1}^T \\ C_{2,k-1} M_{k-1} C_{1,k-1}^T & R_{k-1} + C_{2,k-1} M_{k-1} C_{2,k-1}^T \end{bmatrix} > 0$ .

### 9.3.5.3 Performance

Subtracting from (83) from (68) gives  $\tilde{x}_{k/k} = A_{k-1} \hat{x}_{k-1/k-1} + B_{k-1} w_{k-1} - A_{k-1} \hat{x}_{k-1/k-1} + L_k C_{2,k} A_{k-1} \hat{x}_{k-1/k-1} + L_k (C_{2,k} (A_{k-1} x_{k-1} + B_{k-1} w_{k-1}) + v_k)$ . Denote  $i_k = \begin{bmatrix} v_k \\ w_{k-1} \end{bmatrix}$ , then the filtered error system may be written as

$$\begin{aligned} \begin{bmatrix} \tilde{x}_{k/k} \\ \tilde{y}_{k-1/k-1} \end{bmatrix} &= \begin{bmatrix} (I - L_k C_{2,k}) A_{k-1} & [-L_k & (I - L_k C_{2,k}) B_{k-1}] \\ C_{1,k} & [0 & 0] \end{bmatrix} \begin{bmatrix} \tilde{x}_{k-1/k-1} \\ i_k \end{bmatrix} \\ &= \mathcal{R}_{yi} i, \end{aligned} \quad (87)$$

with  $\tilde{x}_0 = 0$ , where  $\mathcal{R}_{yi} = \begin{bmatrix} (I - L_k C_{2,k}) A_{k-1} & [-L_k & (I - L_k C_{2,k}) B_{k-1}] \\ C_{1,k} & [0 & 0] \end{bmatrix}$ . It is shown below that the filtered error satisfies the desired performance objective.

**Lemma 9 [11], [13], [30]:** In respect of the  $H_\infty$  problem (68) – (70), (82), the solution (83) – (84) achieves the performance  $\sum_{k=0}^{N-1} \tilde{y}_{k/k}^T \tilde{y}_{k/k} - \gamma^2 \sum_{k=0}^{N-1} i_k^T i_k < 0$ .

**Proof:** By applying the Bounded Real Lemma to  $\mathcal{R}_{yi}^H$  and taking the adjoint to address  $\mathcal{R}_{yi}$ , it is required that there exists a positive definite symmetric solution to

$$\begin{aligned}
P_{k+1} &= (I - L_k C_{2,k}) A_{k-1} P_k A_{k-1}^T (I - C_{2,k}^T L_k^T) \\
&+ \gamma^{-2} (I - L_k C_{2,k}) A_{k-1} P_k C_{1,k}^T (I - \gamma^{-2} C_{1,k} P_k C_{1,k}^T)^{-1} C_{1,k} P_k A_{k-1}^T (I - L_k C_{2,k})^T \\
&+ (I - L_k C_{2,k}) B_{k-1} Q_{k-1} B_{k-1}^T (I - C_{2,k}^T L_k^T) + L_k R_k L_k^T \\
&= (I - L_k C_{2,k}) A_{k-1} (P_k^{-1} - \gamma^{-2} C_{1,k}^T C_{1,k}) A_{k-1}^T (I - C_{2,k}^T L_k^T) \\
&+ (I - L_k C_{2,k}) B_{k-1} Q_{k-1} B_{k-1}^T (I - C_{2,k}^T L_k^T) + L_k R_k L_k^T,
\end{aligned} \tag{88}$$

in which use was made of the Matrix Inversion Lemma. Defining

$$P_{k/k-1}^{-1} = P_k^{-1} - \gamma^{-2} C_{1,k}^T C_{1,k}, \tag{89}$$

using (85) and applying the Matrix Inversion Lemma leads to

$$\begin{aligned}
P_{k+1/k}^{-1} + \gamma^{-2} C_{1,k}^T C_{1,k} &= (I - L_k C_{2,k}) (A_{k-1} P_{k/k-1} A_{k-1}^T + B_{k-1} Q_{k-1} B_{k-1}^T) (I - C_{2,k}^T L_k^T) + L_k R_k L_k^T \\
&= (I - L_k C_{2,k}) M_k (I - C_{2,k}^T L_k^T) + L_k R_k L_k^T \\
&= M_k - M_k C_{2,k}^T (R_k + C_{2,k} M_k C_{2,k}^T)^{-1} C_{2,k} M_k \\
&= (M_k^{-1} + C_{2,k}^T R_k^{-1} C_{2,k})^{-1},
\end{aligned} \tag{90}$$

where

$$M_k = A_{k-1} P_{k/k-1} A_{k-1}^T + B_{k-1} Q_{k-1} B_{k-1}^T. \tag{91}$$

It follows from (90) that  $P_{k+1/k}^{-1} + \gamma^{-2} C_{1,k}^T C_{1,k} = M_k^{-1} + C_{2,k}^T R_k^{-1} C_{2,k}$  and

$$\begin{aligned}
P_{k/k-1}^{-1} &= M_{k-1}^{-1} + C_{2,k-1}^T R_{k-1}^{-1} C_{2,k-1} - \gamma^{-2} C_{1,k-1}^T C_{1,k-1} \\
&= M_{k-1}^{-1} + \bar{C}_{k-1}^T \bar{R}_{k-1}^{-1} \bar{C}_{k-1},
\end{aligned} \tag{92}$$

where  $\bar{C}_k = \begin{bmatrix} C_{1,k} \\ C_{2,k} \end{bmatrix}$  and  $\bar{R}_k = \begin{bmatrix} -\gamma^{-2} I & 0 \\ 0 & R_k \end{bmatrix}$ . Substituting (92) into (91) yields

$$M_k = A_{k-1} (M_{k-1}^{-1} + \bar{C}_{k-1}^T \bar{R}_{k-1}^{-1} \bar{C}_{k-1})^{-1} A_{k-1}^T + B_{k-1} Q_{k-1} B_{k-1}^T, \tag{93}$$

which is the same as (86). From (77), the existence of  $M_k > 0$  for the above Riccati difference equation implies the existence of a  $P_k > 0$  for (88). Thus, it follows from Lemma 7 that the stated performance objective is achieved.

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“I can live with doubt and uncertainty. I think it’s much more interesting to live not knowing than to have answers which might be wrong.” *Richard Phillips Feynman*

### 9.3.6 Solution to the General Filtering Problem

Limebeer, Green and Walker express Riccati difference equations such as (86) in a compact form using J-factorisation [5], [21]. The solutions for the general filtering problem follow immediately from their results. Consider

$$\begin{bmatrix} x_{k+1} \\ \tilde{y}_{k/k} \\ z_k \end{bmatrix} = \begin{bmatrix} A_k & B_{1,1,k} & 0 \\ C_{1,1,k} & D_{1,1,k} & D_{1,2,k} \\ C_{2,1,k} & D_{2,1,k} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ i_k \\ -\hat{y}_{1,k/k} \end{bmatrix}. \quad (94)$$

Let  $J_k = \begin{bmatrix} E\{i_k i_k^T\} & 0 \\ 0 & -\gamma^2 \end{bmatrix}$ ,  $\bar{C}_k = \begin{bmatrix} C_{1,1,k} \\ C_{2,1,k} \end{bmatrix}$ ,  $\bar{D}_k = \begin{bmatrix} D_{1,1,k} & D_{1,2,k} \\ D_{2,1,k} & 0 \end{bmatrix}$  and  $\bar{B}_k = \begin{bmatrix} B_{1,1,k} & 0 \end{bmatrix}$ . From the approach of [5], [21], the Riccati difference equation corresponding to the  $H_\infty$  problem (94) is

$$M_{k+1} = A_k M_k A_k^T + \bar{B}_k J_k \bar{B}_k^T - (A_k M_k \bar{C}_k^T + \bar{B}_k J_k \bar{D}_k^T)(\bar{C}_k M_k \bar{C}_k^T + \bar{D}_k J_k \bar{D}_k^T)^{-1}(A_k M_k \bar{C}_k^T + \bar{B}_k J_k \bar{D}_k^T)^T. \quad (95)$$

Suppose in a general filtering problem that  $\mathcal{G}_2$  is realised by (68),  $y_{2,k} = C_{2,k}x_k + D_{2,k}w_k$ ,  $\mathcal{G}_1$  is realised by (68) and  $y_{1,k} = C_{1,k}x_k + D_{1,k}w_k$ . Then substituting  $B_{1,1,k} = \begin{bmatrix} 0 & B_k \end{bmatrix}$ ,  $C_{1,1,k} = C_{1,k}$ ,  $C_{2,1,k} = C_{2,k}$ ,  $D_{1,1,k} = \begin{bmatrix} 0 & D_{1,k} \end{bmatrix}$ ,  $D_{1,2,k} = I$  and  $D_{2,1,k} = \begin{bmatrix} I & D_{2,k} \end{bmatrix}$  into (95) yields

$$\begin{aligned} M_{k+1} &= A_k M_k A_k^T - \begin{bmatrix} A_k M_k C_{1,k}^T + B_k Q_k D_{1,k}^T \\ A_k M_k C_{2,k}^T + B_k Q_k D_{2,k}^T \end{bmatrix} \\ &\quad \times \begin{bmatrix} C_{1,k} M_k C_{1,k}^T + D_{1,k} Q_k D_{1,k}^T - \gamma^2 I & C_{1,k} M_k C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T \\ C_{2,k} M_k C_{1,k}^T + D_{2,k} Q_k D_{1,k}^T & R_k + C_{2,k} M_k C_{2,k}^T + D_{2,k} Q_k D_{2,k}^T \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} C_{1,k} M_k A_k^T + D_{1,k} Q_k B_{1,k}^T & C_{2,k} M_k A_k^T + D_{2,k} Q_k B_{2,k}^T \end{bmatrix} + B_k Q_k B_k^T. \end{aligned} \quad (96)$$

The filter solution is given by

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k-1} + K_k (z_k - C_{2,1,k} \hat{x}_{k/k-1}), \quad (97)$$

$$\hat{y}_{1,k/k} = C_{1,1,k} \hat{x}_{k/k-1} + L_k (z_k - C_{2,1,k} \hat{x}_{k/k-1}), \quad (98)$$



where  $K_k = (A_k M_k C_{2,k}^T + B_k Q_k D_{2,k}^T) \Omega_k^{-1}$ ,  $L_k = (C_{1,k} M_k C_{2,k}^T + D_{1,k} Q_k D_{2,k}^T) \Omega_k^{-1}$  and  $\Omega_k = C_{2,k} M_k C_{2,k}^T + D_{2,k} Q_k D_{2,k}^T + R_k$ .

### 9.3.7 Discrete-Time $H_\infty$ Smoothing

#### 9.3.7.1 Problem Definition

Suppose that measurements (72) of a system (68) – (69) are available over an interval  $k \in [1, N]$ . The problem of interest is to calculate smoothed estimates  $\hat{y}_{k/N}$  of  $y_k$  such that the error sequence

$$\tilde{y}_{k/N} = y_k - \hat{y}_{k/N} \quad (99)$$

is in  $\ell_2$ .

#### 9.3.7.2 $H_\infty$ Solution

The following fixed-interval smoother for output estimation [28] employs the gain for the  $H_\infty$  predictor,

$$K_k = A_k P_{k/k-1} C_{2,k}^T \Omega_k^{-1}, \quad (100)$$

where  $\Omega_k = C_{2,k} P_{k/k-1} C_{2,k}^T + R_k$ , in which  $P_{k/k-1}$  is obtained from (76) and (77). The gain (100) is used in the minimum-variance smoother structure described in Chapter 7, viz.,

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \alpha_k \end{bmatrix} = \begin{bmatrix} A_k - K_k C_{2,k} & K_k \\ -\Omega_k^{-1/2} C_{2,k} & \Omega_k^{-1/2} \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix}, \quad (101)$$

$$\begin{bmatrix} \xi_{k-1} \\ \beta_k \end{bmatrix} = \begin{bmatrix} A_k^T - C_{2,k}^T K_k^T & C_{2,k}^T \Omega_k^{-1/2} \\ -K_k^T & \Omega_k^{-1/2} \end{bmatrix} \begin{bmatrix} \xi_k \\ \alpha_k \end{bmatrix}, \xi_N = 0, \quad (102)$$

$$\hat{y}_{k/N} = z_k - R_k \beta_k. \quad (103)$$

It is argued below that this smoother meets the desired  $H_\infty$  performance objective.

#### 9.3.7.3 $H_\infty$ Performance

It is easily shown that the smoother error system is

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“I have had my results for a long time; but I do not yet know how I am to arrive at them.” *Karl Friedrich Gauss*

$$\begin{aligned} \begin{bmatrix} \tilde{x}_{k+1/k} \\ \lambda_{k-1} \\ \tilde{y}_{k/N} \end{bmatrix} &= \begin{bmatrix} A_k - K_k C_{2,k} & 0 \\ C_{2,k}^T \Omega_k^{-1} C_{2,k} & A_k^T - C_{2,k}^T K_k^T \\ R_k \Omega_k^{-1} C_{2,k} & -R_k K_k^T \end{bmatrix} \begin{bmatrix} -K_k & B_k \\ C_{2,k}^T \Omega_k^{-1} & 0 \\ R_k \Omega_k^{-1} - I & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_{k/k-1} \\ \lambda_k \\ v_k \\ w_k \end{bmatrix}, \\ &= \mathcal{R}_{\tilde{y}_i} i, \end{aligned} \quad (104)$$

with  $\tilde{x}_0 = 0$ , where  $\tilde{x}_{k/k-1} = x_k - \hat{x}_{k/k-1}$ ,  $i = \begin{bmatrix} v \\ w \end{bmatrix}$  and

$$\mathcal{R}_{\tilde{y}_i} = \begin{bmatrix} A_k - K_k C_{2,k} & 0 \\ C_{2,k}^T \Omega_k^{-1} C_{2,k} & A_k^T - C_{2,k}^T K_k^T \\ R_k \Omega_k^{-1} C_{2,k} & -R_k K_k^T \end{bmatrix} \begin{bmatrix} -K_k & B_k \\ C_{2,k}^T \Omega_k^{-1} & 0 \\ R_k \Omega_k^{-1} - I & 0 \end{bmatrix}.$$

**Lemma 10:** *In respect of the smoother error system (104), if there exists a symmetric positive definite solutions to (77) for  $\gamma > 0$ , then the smoother (101) – (103) achieves  $\mathcal{R}_{\tilde{y}_i} \in \ell_\infty$ , that is,  $i \in \ell_2$  implies  $e \in \ell_2$ .*

**Outline of Proof:** From Lemma 8,  $\tilde{x} \in \ell_2$ , since it evolves within the predictor error system. Therefore,  $\lambda \in \ell_2$ , since it evolves within the adjoint predictor error system. Then  $\hat{y} \in \ell_2$ , since it is a linear combination of  $\tilde{x}$ ,  $\lambda$  and  $i \in \ell_2$ .

### 9.3.7.4 Performance Comparison

**Example 4 [28].** A voiced speech utterance “a e i o u” was sampled at 8 kHz for the purpose of comparing smoother performance. Simulations were conducted with the zero-mean, unity-variance speech sample interpolated to a 16 kHz sample rate, to which 200 realisations of Gaussian measurement noise were added and the signal to noise ratio was varied from -5 to 5 dB. The speech sample is modelled as a first-order autoregressive process

$$x_{k+1} = Ax_k + w_k, \quad (105)$$

where  $A \in \mathbb{R}$ ,  $0 < A < 1$ . Estimates for  $\sigma_w^2$  and  $A$  were calculated at 20 dB SNR using an EM algorithm, see Chapter 8.

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“If I have seen further it is only by standing on the shoulders of giants.” *Isaac Newton*

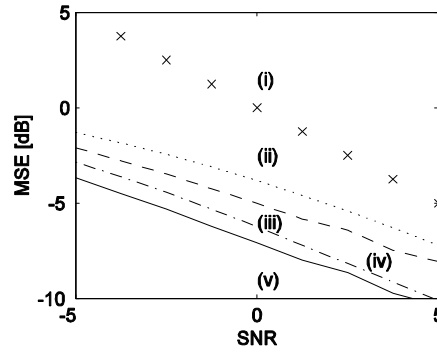


Fig. 9. Speech estimate performance comparison: (i) data (crosses), (ii) Kalman filter (dotted line), (iii)  $H_\infty$  filter (dashed line), (iv) minimum-variance smoother (dot-dashed line) and (v)  $H_\infty$  smoother (solid line).

Simulations were conducted in which a minimum-variance filter and a fixed-interval smoother were employed to recover the speech message from noisy measurements. The results are provided in Fig. 9. As expected, the smoother outperforms the filter. Searches were conducted for minimum values of  $\gamma$  such that solutions to the design Riccati difference equations were positive definite for each noise realisation. The performance of the resulting  $H_\infty$  filter and smoother are indicated by the dashed line and solid line of the figure. It can be seen for this example that the  $H_\infty$  filter outperforms the Kalman filter. The figure also indicates that the robust smoother provides the best performance and exhibits about 4 dB reduction in mean-square-error compared to the Kalman filter at 0 dB SNR. This performance benefit needs to be reconciled against the extra calculation cost of combining robust forward and backward state predictors within (101) – (103).

### 9.3.7.5 High SNR and Low SNR Asymptotes

An understanding of why robust solutions are beneficial in the presence of uncertainties can be gleaned by examining single-input-single-output filtering and equalisation. Consider a time-invariant plant having the canonical form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & & \ddots & 0 \\ \vdots & & & \\ -a_0 & -a_1 & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_2 = [c \quad 0 \quad \cdots \quad 0],$$

$a_0, \dots, a_{n-1}, c \in \mathbb{R}$ . Since the plant is time-invariant, the transfer function exists and is denoted by  $G(z)$ . Some notation is defined prior to stating some observations for output estimation problems. Suppose that an  $H_\infty$  filter has been constructed for the

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“In computer science, we stand on each other's feet.” *Brian K. Reid*

above plant. Let the  $H_\infty$  algebraic Riccati equation solution, predictor gain, filter gain, predictor, filter and smoother transfer function matrices be denoted by  $P^{(\infty)}$ ,  $K^{(\infty)}$ ,  $L^{(\infty)}$ ,  $H_p^{(\infty)}(z)$ ,  $H_F^{(\infty)}(z)$  and  $H_S^{(\infty)}(z)$  respectively. The  $H_\infty$  filter transfer function matrix may be written as  $H_F^{(\infty)}(z) = L^{(\infty)} + (I - L^{(\infty)})H_p^{(\infty)}(z)$  where  $L^{(\infty)} = I - R(\Omega^{(\infty)})^{-1}$ . The transfer function matrix of the map from the inputs to the filter output estimation error is

$$R_{ji}^{(\infty)}(z) = [H_F^{(\infty)}(z)\sigma_v \quad (H_F^{(\infty)}(z) - I)G(z)\sigma_w]. \quad (106)$$

The  $H_\infty$  smoother transfer function matrix can be written as  $H_S^{(\infty)}(z) = I - R(I - (H_p^{(\infty)}(z))^H)(\Omega^{(\infty)})^{-1}(I - H_p^{(\infty)}(z))$ . Similarly, let  $P^{(2)}$ ,  $K^{(2)}$ ,  $L^{(2)}$ ,  $H_F^{(2)}(z)$  and  $H_S^{(2)}(z)$  denote the minimum-variance algebraic Riccati equation solution, predictor gain, filter gain, filter and smoother transfer function matrices respectively.

**Proposition 1 [28]:** *In the above output estimation problem:*

(i)

$$\lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_F^{(\infty)}(e^{j\omega})| = 1. \quad (107)$$

(ii)

$$\lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_F^{(2)}(e^{j\omega})| < \lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_F^{(\infty)}(e^{j\omega})|. \quad (108)$$

(iii)

$$\lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |R_{ei}^{(\infty)}(R_{ei}^{(\infty)})^H(e^{j\omega})| = \lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_F^{(\infty)}(H_F^{(\infty)})^H(e^{j\omega})|\sigma_v^2. \quad (109)$$

(iv)

$$\lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_S^{(\infty)}(e^{j\omega})| = 1. \quad (110)$$

(v)

$$\lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_S^{(2)}(e^{j\omega})| < \lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_S^{(\infty)}(e^{j\omega})|. \quad (111)$$

**Outline of Proof:** (i) Let  $p_{(1,1)}^{(\infty)}$  denote the  $(1,1)$  component of  $P^{(\infty)}$ . The low measurement noise observation (107) follows from  $L^{(\infty)} = 1 - \sigma_v^2(c^2 p_{(1,1)}^{(\infty)} + \sigma_v^2)^{-1}$  which implies  $\lim_{\sigma_v^2 \rightarrow 0} L^{(\infty)} = 1$ .

(ii) Observation (108) follows from  $p_{(1,1)}^{(\infty)} > p_{(1,1)}^{(2)}$ , which results in  $\lim_{\sigma_v^2 \rightarrow 0} L^{(\infty)} > \lim_{\sigma_v^2 \rightarrow 0} L^{(2)}$ .

(iii) Observation (109) follows immediately from the application of (107) in (106).

(iv) Observation (110) follows from  $\lim_{\sigma_v^2 \rightarrow 0} \sigma_v^2 (c^2 p_{(1,1)}^{(\infty)} + \sigma_v^2)^{-1} = 0$ .

(v) Observation (111) follows from  $p_{(1,1)}^{(\infty)} > p_{(1,1)}^{(2)}$  which results in  $\lim_{\sigma_v^2 \rightarrow 0} \Omega_3^{(\infty)} > \lim_{\sigma_v^2 \rightarrow 0} \Omega^{(2)}$ .

An interpretation of (107) and (110) is that the maximum magnitudes of the filters and smoothers asymptotically approach a short circuit (or zero impedance) when  $\sigma_v^2 \rightarrow 0$ . From (108) and (111), as  $\sigma_v^2 \rightarrow 0$ , the maximum magnitudes of the  $H_\infty$  solutions approach the short circuit asymptote closer than the optimal minimum-variance solutions. That is, for low measurement noise, the robust solutions accommodate some uncertainty by giving greater weighting to the data. Since  $\lim_{\sigma_v^2 \rightarrow 0} \|R_{\hat{y}_i}\|_\infty \rightarrow \sigma_v$  and the  $H_\infty$  filter achieves the performance  $\|R_{\hat{y}_i}\|_\infty < \gamma$ , it

follows from (109) that an *a priori* design estimate is  $\gamma = \sigma_v$ .

Suppose now that a time-invariant plant has the transfer function  $G(z) = C(zI - A)^{-1}B + D$ , where  $A$ ,  $B$  and  $C$  are defined above together with  $D \in \mathbb{R}$ . Consider an input estimation (or equalisation) problem in which the transfer function matrix of the causal  $H_\infty$  solution that estimates the input of the plant is

$$H_F^{(\infty)}(z) = QD^T \left( \Omega^{(\infty)} \right)^{-1} - QD^T \left( \Omega^{(\infty)} \right)^{-1} H_p^{(\infty)}(z). \quad (112)$$

The transfer function matrix of the map from the inputs to the input estimation error is

$$R_{\hat{y}_i}^{(\infty)}(z) = [H_F^{(\infty)}(z)\sigma_v \quad (H_F^{(\infty)}G(z) - I)\sigma_w]. \quad (113)$$

The noncausal  $H_\infty$  transfer function matrix of the input estimator can be written as  $H_S^{(\infty)}(z) = QG^{-H}(z)(I - (H_p^{(\infty)}(z))^H)(\Omega_3^{(\infty)})^{-1}(I - H_p^{(\infty)}(z))$ .

**Proposition 2 [28]:** For the above input estimation problem:

(i)

$$\lim_{\sigma_v^2 \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_F^{(\infty)}(e^{j\omega})| = 0. \quad (114)$$

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“Always code as if the guy who ends up maintaining your code will be a violent psychopath who knows where you live.” *Damian Conway*

$$(ii) \quad \lim_{\sigma_v^{-2} \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_F^{(2)}(e^{j\omega})| > \lim_{\sigma_v^{-2} \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_F^{(\infty)}(e^{j\omega})|. \quad (115)$$

$$(iii) \quad \begin{aligned} & \lim_{\sigma_v^{-2} \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |R_{ei}^{(\infty)}(R_{ei}^{(\infty)})^H(e^{j\omega})| \\ &= \lim_{\sigma_v^{-2} \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |(H_F^{(\infty)}G^{(\infty)}(e^{j\omega}) - 1)(H_F^{(\infty)}G^{(\infty)}(e^{j\omega}) - 1)^H| \sigma_w^2. \end{aligned} \quad (116)$$

$$(iv) \quad \lim_{\sigma_v^{-2} \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_S^{(\infty)}(e^{j\omega})| = 0. \quad (117)$$

$$(v) \quad \lim_{\sigma_v^{-2} \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_S^{(2)}(e^{j\omega})| > \lim_{\sigma_v^{-2} \rightarrow 0} \sup_{\omega \in \{-\pi, \pi\}} |H_S^{(\infty)}(e^{j\omega})|. \quad (118)$$

**Outline of Proof:** (i) and (iv) The high measurement noise observations (114) and (117) follow from  $\Omega^{(\infty)} = c^2 p_{(1,1)}^{(\infty)} + d\sigma_w^2 + \sigma_v^2$  which implies  $\lim_{\sigma_v^{-2} \rightarrow 0} (\Omega_3^{(\infty)})^{-1} = 0$ .

(ii) and (v) The observations (115) and (118) follow from  $p_{(1,1)}^{(\infty)} > p_{(1,1)}^{(2)}$ , which results in  $\lim_{\sigma_v^{-2} \rightarrow 0} \Omega_3^{(\infty)} > \lim_{\sigma_v^{-2} \rightarrow 0} \Omega^{(2)}$ .

(iii) The observation (116) follows immediately from the application of (114) in (113).

An interpretation of (114) and (117) is that the maximum magnitudes of the equalisers asymptotically approach an open circuit (or infinite impedance) when  $\sigma_v^{-2} \rightarrow 0$ . From (115) and (118), as  $\sigma_v^{-2} \rightarrow 0$ , the maximum magnitude of the  $H_\infty$  solution approaches the open circuit asymptote closer than that of the optimum minimum-variance solution. That is, under high measurement noise conditions, robust solutions accommodate some uncertainty by giving less weighting to the data. Since  $\lim_{\sigma_v^{-2} \rightarrow 0} \|R_{\hat{y}i}\|_\infty \rightarrow \sigma_w$ , the  $H_\infty$  solution achieves the performance

$\|R_{\hat{y}i}\|_\infty < \gamma$ , it follows from (116) than an *a priori* design estimate is  $\gamma = \sigma_w$ .

Proposition 1 follows intuitively. Indeed, the short circuit asymptote is sometimes referred to as the singular filter. Proposition 2 may appear counter-intuitive and warrants further explanation. When the plant is minimum phase and the measurement noise is negligible, the equaliser inverts the plant. Conversely, when the equalisation problem is dominated by measurement noise, the solution is a low

gain filter; that is, the estimation error is minimised by giving less weighting to the data.

## 9.4 Chapter Summary

Uncertainties are invariably present within the specification of practical problems. Consequently, robust solutions have arisen to accommodate uncertain inputs and plant models. The  $H_\infty$  performance objective is to minimise the ratio of the output energy to the input energy of an error system, that is, minimise

$$\sup_{\|i\|_2 \neq 0} \|\mathcal{R}_{\hat{y}}\|_\infty = \sup_{\|i\|_2 \neq 0} \frac{\|\hat{y}\|_2}{\|i\|_2} \leq \gamma$$

for some  $\gamma \in \mathbb{R}$ . In the time-invariant case, the objective is equivalent to minimising the maximum magnitude of the error power spectrum density.

Predictors, filters and smoothers that satisfy the above performance objective are found by applying the Bounded Real Lemma. The standard solution structures are retained but larger design error covariances are employed to account for the presence of uncertainty. In continuous time output estimation, the error covariance is found from the solution of

$$\begin{aligned} \dot{P}(t) = & A(t)P(t) + P(t)A^T(t) + B(t)Q(t)B^T(t) \\ & - P(t)(C^T(t)R^{-1}(t)C(t) - \gamma^{-2}C^T(t)C(t))P(t). \end{aligned}$$

Discrete-time predictors, filters and smoothers for output estimation rely on the solution of

$$\begin{aligned} P_{k+1} = & A_k P_k A_k^T + B_k Q_k B_k^T \\ & - A_k P_k \begin{bmatrix} C_k^T & C_k^T \end{bmatrix} \begin{bmatrix} C_k P_k C_k^T - \gamma^2 I & C_k P_k C_k^T \\ C_k P_k C_k^T & R_k + C_k P_k C_k^T \end{bmatrix}^{-1} \begin{bmatrix} C_k \\ C_k \end{bmatrix} P_k A_k^T. \end{aligned}$$

It follows that the  $H_\infty$  designs revert to the optimum minimum-variance solutions as  $\gamma^2 \rightarrow 0$ . Since robust solutions are conservative, the art of design involves finding satisfactory trade-offs between average and worst-case performance criteria, namely, tweaking the  $\gamma$ .

A summary of suggested approaches for different linear estimation problem conditions is presented in Table 1. When the problem parameters are known precisely then the optimum minimum-variance solutions cannot be improved upon. However, when the inputs or the models are uncertain, robust solutions may provide improved mean-square-error performance. In the case of low measurement noise output-estimation, the benefit arises because greater weighting

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“A computer lets you make more mistakes than almost any invention in history, with the possible exceptions of tequila and hand guns.” *Mitch Ratcliffe*

is given to the data. Conversely, for high measurement noise input estimation, robust solutions accommodate uncertainty by giving less weighting to the data.

PROBLEM CONDITIONS	SUGGESTED APPROACHES
Gaussian process and measurement noises, known 2 <sup>nd</sup> -order statistics. Known system model parameters.	<ol style="list-style-type: none"> <li>1. Optimal minimum-variance (or Kalman) filter.</li> <li>2. Fixed-lag smoothers, which improve on filter performance (see Lemma 3 and Example 1 of Chapter 7). They suit on-line applications and have low additional complexity. A sufficiently large smoothing lag results in optimal performance (see Example 3 of Chapter 7).</li> <li>3. Maximum-likelihood (or Rauch-Tung-Striebel) smoothers, which also improve on filter performance (see Lemma 6 of Chapter 6 and Lemma 4 of Chapter 7). They can provide close to optimal performance (see Example 5 of Chapter 6).</li> <li>4. The minimum-variance smoother provides the best performance (see Lemma 12 of Chapter 6 and Lemma 8 of Chapter 7) at the cost of increased complexity (see Example 5 of Chapter 6 and Example 2 of Chapter 7).</li> </ol>
Uncertain process and measurement noises, known 2 <sup>nd</sup> -order statistics. Known system model parameters.	<ol style="list-style-type: none"> <li>1. Optimal minimum-variance filter, which does not rely on Gaussian noise assumptions.</li> <li>2. Optimal minimum-variance smoother, which similarly does not rely on Gaussian noise assumptions (see Example 6 of Chapter 6).</li> <li>3. Robust filter which trades off <math>H_\infty</math> performance (see Lemmas 2, 9) and mean-square-error performance (see Example 3).</li> <li>4. Robust smoother which trades off <math>H_\infty</math> performance (see Lemmas 5, 10) and mean-square-error performance (see Example 3).</li> </ol>
Uncertain processes and measurement noises. Uncertain system model parameters.	<ol style="list-style-type: none"> <li>1. Robust filter (see Example 4).</li> <li>2. Robust smoother (see Example 4).</li> <li>3. Robust filter or smoother with scaled inputs (see Lemma 3).</li> </ol>

Table 1. Suggested approaches for different linear estimation problem conditions.<sup>35</sup>

## 9.5 Problems

**Problem 1** [31].

- (i) Consider a system  $\mathcal{G}$  having the state-space representation  $\dot{x}(t) = Ax(t) + Bw(t)$ ,  $y(t) = Cx(t)$ . Show that if there exists a matrix  $P = P^T > 0$  such that 
$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} < 0$$
 then 
$$x^T(0)Px(0) + \int_0^T y^T(t)y(t)dt \leq \gamma^2 \int_0^T w^T(t)w(t) dt.$$
- (ii) Generalise (i) for the case where  $y(t) = Cx(t) + Dw(t)$ .

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“The factory of the future will only have two employees, a man and a dog. The man will be there to feed the dog. The dog will be there to keep the man from touching the equipment.” *Alice Kahn*



**Problem 2.** Consider a system  $\mathcal{G}$  modelled by  $\dot{x}(t) = A(t)x(t) + B(t)w(t)$ ,  $y(t) = C(t)x(t) + D(t)w(t)$ . Suppose that the Riccati differential equation

$$-\dot{P}(t) = P(t)(A(t) + B(t)M^{-1}(t)D^T(t)C(t)) + (A(t) + B(t)M^{-1}(t)D^T(t)C(t))^T P(t) + \gamma^{-2}B(t)M^{-1}(t)B^T(t) + C^T(t)(I + D(t)M^{-1}(t)D^T(t))C(t),$$

$M(t) = \gamma^2 I - D^T(t)D(t) > 0$ , has a solution on  $[0, T]$ . Show that  $\|\mathcal{G}\|_\infty \leq \gamma$  for any  $w \in \mathcal{L}_2$ . (Hint: define  $V(x(t)) = x^T(t)P(t)x(t)$  and show that  $\dot{V}(x(t)) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) < 0$ .)

**Problem 3.** For measurements  $z(t) = y(t) + v(t)$  of a system realised by  $\dot{x}(t) = A(t)x(t) + B(t)w(t)$ ,  $y(t) = C(t)x(t)$ , show that the map from the inputs  $i = \begin{bmatrix} v \\ w \end{bmatrix}$  to the  $H_\infty$  fixed-interval smoother error  $e(t|T)$  is

$$\begin{bmatrix} \hat{x}(t) \\ -\xi(t) \\ \hat{y}(t|T) \end{bmatrix} = \begin{bmatrix} A(t) - K(t)C(t) & 0 & B(t) & -K(t) \\ -C^T(t)R^{-1}(t)C(t) & A^T(t) - C^T(t)K^T(t) & 0 & -C^T(t)R^{-1}(t) \\ C(t) & R(t)K^T(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \xi(t) \\ w(t) \\ v(t) \end{bmatrix}.$$

**Problem 4.**

- (i) For a  $\mathcal{G}$  modelled by  $x_{k+1} = A_k x_k + B_k w_k$ ,  $y_k = C_k x_k + D_k w_k$ , show that the existence of a solution to the Riccati difference equation

$$P_k = A_k^T P_{k+1} A_k + \gamma^{-2} A_k^T P_{k+1} B_k (I - \gamma^{-2} B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k + C_k^T C_k$$

is sufficient for  $x_k^T P_k x_k - x_{k+1}^T P_{k+1} x_{k+1} + y_k^T y_k - \gamma^2 w_k^T w_k < 0$ . Hint: construct  $x_{k+1}^T P_{k+1} x_{k+1} - x_k^T P_k x_k$  and show that

$$\begin{aligned} & x_{k+1}^T P_{k+1} x_{k+1} - x_k^T P_k x_k + y_k^T y_k - \gamma^2 w_k^T w_k \\ &= -\gamma^{-2} p_k^T (I - \gamma^{-2} B_k^T P_{k+1} B_k)^{-1} p_k - x_k^T A_k^T P_{k+1} A_k x_k, \end{aligned}$$

where  $p_k = w_k - \gamma^{-2} (I - \gamma^{-2} B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k x_k$ .

- (ii) Show that  $-x_0^T P_0 x_0 + \sum_{k=0}^{N-1} y_k^T y_k - \gamma^2 \sum_{k=0}^{N-1} w_k^T w_k < 0$ .

**Problem 5.** Now consider the model  $x_{k+1} = A_k x_k + B_k w_k$ ,  $y_k = C_k x_k + D_k w_k$  and show that the existence of a solution to the Riccati difference equation

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“On two occasions I have been asked, ‘If you put into the machine wrong figures, will the right answers come out?’ I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.” *Charles Babbage*

$$P_k = A_k^T P_{k+1} A_k + C_k^T C_k + \gamma^{-2} (A_k^T P_{k+1} B_k + C_k^T D_k) (I - \gamma^{-2} B_k^T P_{k+1} B_k - \gamma^{-2} D_k^T D_k)^{-1} (B_k^T P_{k+1} A_k + D_k^T C_k)$$

is sufficient for  $x_k^T P_k x_k - x_{k+1}^T P_{k+1} x_{k+1} + y_k^T y_k - \gamma^2 w_k^T w_k < 0$ . Hint: define

$$P_k = w_k - \gamma^{-2} (I - \gamma^{-2} B_k^T P_{k+1} B_k - \gamma^{-2} D_k^T D_k)^{-1} (B_k^T P_{k+1} A_k + D_k^T C_k).$$

**Problem 6.** Suppose that a predictor attains a  $H_\infty$  performance objective, that is, the conditions of Lemma 8 are satisfied. Show that using the predicted states to construct filtered output estimates  $\hat{y}_{k/k}$  results in  $\tilde{y}_{k/k} = y - \hat{y}_{k/k} \in \ell_2$ .

## 9.6 Glossary

$\mathcal{L}_\infty$	The Lebesgue $\infty$ -space defined as the set of continuous-time systems having finite $\infty$ -norm.
$\mathcal{R}_{yi} \in \mathcal{L}_\infty$	The map $\mathcal{R}_{yi}$ from the inputs $i(t)$ to the estimation error $\tilde{y}(t)$ satisfies $\int_0^T \tilde{y}^T(t) \tilde{y}(t) dt - \gamma^2 \int_0^T i^T(t) i(t) dt < 0$ . Therefore, $i \in \mathcal{L}_2$ implies $\tilde{y} \in \mathcal{L}_2$ .
$\ell_\infty$	The Lebesgue $\infty$ -space defined as the set of discrete-time systems having finite $\infty$ -norm.
$\mathcal{R}_{yi} \in \ell_\infty$	The map $\mathcal{R}_{ei}$ from the inputs $i_k$ to the estimation error $\tilde{y}_k$ satisfies $\sum_{k=0}^{N-1} \tilde{y}_k^T \tilde{y}_k - \gamma^2 \sum_{k=0}^{N-1} i_k^T i_k < 0$ . Therefore, $i \in \ell_2$ implies $\tilde{y} \in \ell_2$ .

## 9.7 References

- [1] J. Stelling, U. Sauer, Z. Szallasi, F. J. Doyle and J. Doyle, “Robustness of Cellular Functions”, *Cell*, vol. 118, pp. 675 – 685, Dec. 2004.
- [2] P. P. Vaidyanathan, “The Discrete-Time Bounded-Real Lemma in Digital Filtering”, *IEEE Transactions on Circuits and Systems*, vol. 32, no. 9, pp. 918 – 924, Sep. 1985.
- [3] I. Petersen, “A Riccati equation approach to the design of stabilizing controllers and observers for a class of uncertain linear systems”, *IEEE Transactions on Automatic Control*, vol. 30, iss. 9, pp. 904 – 907, 1985.
- [4] J. C. Doyle, K. Glover, P. P. Khargonekar and B. A. Francis, “State-Space Solutions to Standard  $H_2$  and  $H_\infty$  Control Problems”, *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831 – 847, Aug. 1989.
- [5] D. J. N. Limebeer, M. Green and D. Walker, “Discrete-Time  $H_\infty$  Control,

---

“Never trust a computer you can’t throw out a window.” *Stephen Gary Wozniak*

- Proceedings 28<sup>th</sup> IEEE Conference on Decision and Control*, Tampa, Florida, pp. 392 – 396, Dec. 198
- [6] T. Basar, “Optimum performance levels for minimax filters, predictors and smoothers” *Systems and Control Letters*, vol. 16, pp. 309 – 317, 198
  - [7] D. S. Bernstein and W. M. Haddad, “Steady State Kalman Filtering with an  $H_\infty$  Error Bound”, *Systems and Control Letters*, vol. 12, pp. 9 – 16, 198
  - [8] U. Shaked, “ $H_\infty$ -Minimum Error State Estimation of Linear Stationary Processes”, *IEEE Transactions on Automatic Control*, vol. 35, no. 5, pp. 554 – 558, May, 1990.
  - [9] T. Başar and P. Bernhard,  *$H^\infty$ -Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, Series in Systems & Control: Foundations & Applications, Birkhäuser, Boston, 1991.
  - [10] K. M. Nagpal and P. P. Khargonekar, “Filtering and Smoothing in an  $H_\infty$  Setting”, *IEEE Transactions on Automatic Control*, vol. 36, no. 2, pp 152 – 166, Feb. 1991.
  - [11] I. Yaesh and U. Shaked, “ $H_\infty$ -Optimal Estimation – The Discrete Time Case”, *Proceedings of the MTNS*, pp. 261 – 267, Jun. 1991.
  - [12] A. Stoorvogel, *The  $H_\infty$  Control Problem*, Series in Systems and Control Engineering, Prentice Hall International (UK) Ltd, Hertfordshire, 1992.
  - [13] U. Shaked and Y. Theodor, “ $H_\infty$  Optimal Estimation: A Tutorial”, *Proceedings 31<sup>st</sup> IEEE Conference on Decision and Control*, pp. 2278 – 2286, Tucson, Arizona, Dec. 1992.
  - [14] C. E. de Souza, U. Shaked and M. Fu, “Robust  $H_\infty$  Filtering with Parametric Uncertainty and Deterministic Input Signal”, *Proceedings 31<sup>st</sup> IEEE Conference on Decision and Control*, Tucson, Arizona, pp. 2305 – 2310, Dec. 1992.
  - [15] D. J. N. Limebeer, B. D. O. Anderson, P. P. Khargonekar and M. Green, “A Game Theoretic Approach to  $H^\infty$  Control for Time-Varying Systems”, *SIAM Journal on Control and Optimization*, vol. 30, no. 2, pp. 262 – 283, Mar. 1992.
  - [16] I. Yaesh and Shaked, “Game Theory Approach to Finite-Time Horizon Optimal Estimation”, *IEEE Transactions on Automatic Control*, vol. 38, no. 6, pp. 957 – 963, Jun. 1993.
  - [17] B. van Keulen,  *$H_\infty$  Control for Distributed Parameter Systems: A State-Space Approach*, Series in Systems & Control: Foundations & Applications, Birkhäuser, Boston, 1993.
  - [18] L. Xie, C. E. De Souza and Y. Wang, “Robust Control of Discrete Time Uncertain Dynamical Systems”, *Automatica*, vol. 29, no. 4, pp. 1133 – 1137, 1993.
  - [19] Y. Theodore, U. Shaked and C. E. de Souza, “A Game Theory Approach To Robust Discrete-Time  $H_\infty$ -Estimation”, *IEEE Transactions on Signal Processing*, vol. 42, no. 6, pp. 1486 – 1495, Jun. 1994.
  - [20] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, vol. 15, SIAM, Philadelphia, 1994.
  - [21] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1995.
  - [22] S. O. R. Moheimani, A. V. Savkin and I. R. Petersen, “Robust Filtering, Prediction, Smoothing and Observability of Uncertain Systems”, *IEEE*

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“Programming today is a race between software engineers striving to build bigger and better idiot-proof programs, and the Universe trying to produce bigger and better idiots. So far, the Universe is winning.” *Rich Cook*

- Transactions on Circuits and Systems, vol. 45, no. 4, pp. 446 – 457, Apr. 1998.
- [23] J. C. Geromel, “Optimal Linear Filtering Under Parameter Uncertainty”, *IEEE Transactions on Signal Processing*, vol. 47, no. 1, pp. 168 – 175, Jan. 1999
- [24] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, Second Edition, SIAM, Philadelphia, 1999
- [25] B. Hassibi, A. H. Sayed and T. Kailath, *Indefinite-Quadratic Estimation and Control: A Unified Approach to  $H^\infty$  and  $H^2$  Theories*, SIAM, Philadelphia, 1999
- [26] G. A. Einicke and L. B. White, “Robust Extended Kalman Filtering”, *IEEE Transactions on Signal Processing*, vol. 47, no. 9, pp. 2596 – 2599, Sep., 1999
- [27] Z. Wang and B. Huang, “Robust  $H_2/H_\infty$  Filtering for Linear Systems with Error Variance Constraints”, *IEEE Transactions on Signal Processing*, vol. 48, no. 9, pp. 2463 – 2467, Aug. 2000.
- [28] G. A. Einicke, “Optimal and Robust Noncausal Filter Formulations”, *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 1069 - 1077, Mar. 2006.
- [29] A. Saberi, A. A. Stoorvogel and P. Sannuti, *Filtering Theory With Applications to Fault Detection, Isolation, and Estimation*, Birkhäuser, Boston, 2007.
- [30] F. L. Lewis, L. Xie and D. Popa, *Optimal and Robust Estimation: With an Introduction to Stochastic Control Theory*, Second Edition, Series in Automation and Control Engineering, Taylor & Francis Group, LLC, 2008.
- [31] J.-C. Lo and M.-L. Lin, “Observer-Based Robust  $H_\infty$  Control for Fuzzy Systems Using Two-Step Procedure”, *IEEE Transactions on Fuzzy Systems*, vol. 12, no. 3, pp. 350 – 359, Jun. 2004.
- [32] E. Blanco, P. Neveux and H. Thomas, “The  $H_\infty$  Fixed-Interval Smoothing Problem for Continuous Systems”, *IEEE Transactions on Signal Processing*, vol. 54, no. 11, pp. 4085 – 4090, Nov. 2006.
- [33] G. A. Einicke, “A Solution to the Continuous-Time H-infinity Fixed-Interval Smoother Problem”, *IEEE Transactions on Automatic Control*, vol. 54, no. 12, pp. 2904 – 2908, Dec. 2009
- [34] G. A. Einicke, “Asymptotic Optimality of the Minimum-Variance Fixed-Interval Smoother”, *IEEE Transactions on Signal Processing*, vol. 55, no. 4, pp. 1543 – 1547, Apr. 2007.
- [35] G. A. Einicke, J. C. Ralston, C. O. Hargrave, D. C. Reid and D. W. Hainsworth, “Longwall Mining Automation, An Application of Minimum-Variance Smoothing”, *IEEE Control Systems Magazine*, vol. 28, no. 6, pp. 28 – 37, Dec. 2008.
- [36] K. Ogata, *Discrete-time Control Systems*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1987.

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“The most amazing achievement of the computer software industry is its continuing cancellation of the steady and staggering gains made by the computer hardware industry.” *Henry Petroski*

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## 10. Nonlinear Prediction, Filtering and Smoothing

### 10.1 Introduction

The Kalman filter is widely used for linear estimation problems where its behaviour is well-understood. Under prescribed conditions, the estimated states are unbiased and stability is guaranteed. Many real-world problems are nonlinear which requires amendments to linear solutions. If the nonlinear models can be expressed in a state-space setting then the Kalman filter may find utility by applying linearisations at each time step. In the two-dimensional case, linearising means finding tangents to the curves of interest about the current estimates, so that the standard filter recursions can be employed in tandem to produce predictions for the next step. This approach is known as extended Kalman filtering – see [1] – [5].

Extended Kalman filters (EKFs) revert to optimal Kalman filters when the problems become linear. Thus, EKFs can yield approximate minimum-variance estimates. However, there are no accompanying performance guarantees and they fall into the try-at-your-own-risk category. Indeed, Anderson and Moore [3] caution that the EKF “can be satisfactory on occasions”. A number of compounding factors can cause performance degradation. The approximate linearisations may be crude and are carried out about estimated states (as opposed to true states). Observability problems occur when the variables do not map onto each other, giving rise to discontinuities within estimated state trajectories. Singularities within functions can result in non-positive solutions to the design Riccati equations and lead to instabilities.

The discussion includes suggestions for performance improvement and is organised as follows. The next section begins with Taylor series expansions, which are prerequisites for linearisation. First, second and third-order EKFs are then derived. EKFs tend to be prone to instability and a way of enforcing stability is to masquerade the design Riccati equation by a faux version. This faux algebraic Riccati equation technique [6] – [10] is presented in Section 10.3. In Section 10.4, the higher order terms discarded by an EKF are treated as uncertainties. It is

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“It is the mark of an instructed mind to rest satisfied with the degree of precision to which the nature of the subject admits and not to seek exactness when only an approximation of the truth is possible.”  
*Aristotle*

shown that a robust EKF arises by solving a scaled  $H_\infty$  problem in lieu of one possessing uncertainties. Nonlinear smoother procedures can be designed similarly. The use of fixed-lag and Rauch-Tung-Striebel smoothers may be preferable from a complexity perspective. However, the approximate minimum-variance and robust smoothers, which are presented in Section 10.5, revert to optimal solutions when the nonlinearities and uncertainties diminish. Another way of guaranteeing stability is to by imposing constraints and one such approach is discussed in Section 10.6.

## 10.2 Extended Kalman Filtering

### 10.2.1 Taylor Series Expansion

A nonlinear function  $a_k(x): \mathbb{R}^n \rightarrow \mathbb{R}$  having  $n$  continuous derivatives may be expanded as a Taylor series about a point  $x_0$

$$\begin{aligned}
 a_k(x) = & a_k(x_0) + \frac{1}{1!}(x-x_0)^T \nabla a_k(x_0) \\
 & + \frac{1}{2!}(x-x_0)^T \nabla^T \nabla a_k(x_0)(x-x_0) \\
 & + \frac{1}{3!}(x-x_0)^T \nabla^T \nabla(x-x_0) \nabla a_k(x_0)(x-x_0) \\
 & + \frac{1}{4!}(x-x_0)^T \nabla^T \nabla(x-x_0) \nabla(x-x_0) \nabla a_k(x_0)(x-x_0) + \dots,
 \end{aligned} \tag{1}$$

where  $\nabla a_k = \begin{bmatrix} \frac{\partial a_k}{\partial x_1} & \dots & \frac{\partial a_k}{\partial x_n} \end{bmatrix}$  is known as the gradient of  $a_k(\cdot)$  and

$$\nabla^T \nabla a_k = \begin{bmatrix} \frac{\partial^2 a_k}{\partial x_1^2} & \frac{\partial^2 a_k}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 a_k}{\partial x_1 \partial x_n} \\ \frac{\partial^2 a_k}{\partial x_2 \partial x_1} & \frac{\partial^2 a_k}{\partial x_2^2} & \dots & \frac{\partial^2 a_k}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 a_k}{\partial x_n \partial x_1} & \frac{\partial^2 a_k}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 a_k}{\partial x_n^2} \end{bmatrix}$$

is called a Hessian matrix.

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“In the real world, nothing happens at the right place at the right time. It is the job of journalists and historians to correct that.” *Samuel Langhorne Clemens aka. Mark Twain*

## 10.2.2 Nonlinear Signal Models

Consider nonlinear systems having state-space representations of the form

$$x_{k+1} = a_k(x_k) + b_k(x_k)w_k, \quad (2)$$

$$y_k = c_k(x_k), \quad (3)$$

where  $a_k(\cdot)$ ,  $b_k(\cdot)$  and  $c_k(\cdot)$  are continuous differentiable functions. For a scalar function,  $a_k(x) : \mathbb{R} \rightarrow \mathbb{R}$ , its Taylor series about  $x = x_0$  may be written as

$$\begin{aligned} a_k(x) = & a_k(x_0) + (x - x_0) \left. \frac{\partial a_k}{\partial x} \right|_{x=x_0} + \frac{1}{2} (x - x_0)^2 \left. \frac{\partial^2 a_k}{\partial x^2} \right|_{x=x_0} \\ & + \frac{1}{6} (x - x_0)^3 \left. \frac{\partial^3 a_k}{\partial x^3} \right|_{x=x_0} + \cdots + \frac{1}{n!} (x - x_0)^n \left. \frac{\partial^n a_k}{\partial x^n} \right|_{x=x_0}. \end{aligned} \quad (4)$$

Similarly, Taylor series for  $b_k(x) : \mathbb{R} \rightarrow \mathbb{R}$  and  $c_k(x) : \mathbb{R} \rightarrow \mathbb{R}$  about  $x = x_0$  are

$$\begin{aligned} b_k(x) = & b_k(x_0) + (x - x_0) \left. \frac{\partial b_k}{\partial x} \right|_{x=x_0} + \frac{1}{2} (x - x_0)^2 \left. \frac{\partial^2 b_k}{\partial x^2} \right|_{x=x_0} \\ & + \frac{1}{6} (x - x_0)^3 \left. \frac{\partial^3 b_k}{\partial x^3} \right|_{x=x_0} + \cdots + \frac{1}{n!} (x - x_0)^n \left. \frac{\partial^n b_k}{\partial x^n} \right|_{x=x_0}, \end{aligned} \quad (5)$$

and

$$\begin{aligned} c_k(x) = & c_k(x_0) + (x - x_0) \left. \frac{\partial c_k}{\partial x} \right|_{x=x_0} + \frac{1}{2} (x - x_0)^2 \left. \frac{\partial^2 c_k}{\partial x^2} \right|_{x=x_0} \\ & + \frac{1}{6} (x - x_0)^3 \left. \frac{\partial^3 c_k}{\partial x^3} \right|_{x=x_0} + \cdots + \frac{1}{n!} (x - x_0)^n \left. \frac{\partial^n c_k}{\partial x^n} \right|_{x=x_0}, \end{aligned} \quad (6)$$

respectively.

## 10.2.3 First-Order Extended Kalman Filter

Suppose that filtered estimates  $\hat{x}_{k/k}$  of  $x_k$  are desired given observations

$$z_k = c_k(x_k) + v_k, \quad (7)$$

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“You will always define events in a manner which will validate your agreement with reality.” *Steve Anthony Maraboli*



where  $v_k$  is a measurement noise sequence. A first-order EKF for the above problem is developed below. Following the approach within [3], the nonlinear system (2) – (3) is approximated by

$$x_{k+1} = A_k x_k + B_k w_k + \mu_k, \quad (8)$$

$$y_k = C_k x_k + \pi_k, \quad (9)$$

where  $A_k$ ,  $B_k$ ,  $C_k$ ,  $\mu_k$  and  $\pi_k$  are found from suitable truncations of the Taylor series for each nonlinearity. From Chapter 4, a filter for the above model is given by

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - C_k \hat{x}_{k/k-1} - \pi_k), \quad (10)$$

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k} + \mu_k, \quad (11)$$

where  $L_k = P_{k/k-1} C_k^T \Omega_k^{-1}$  is the filter gain, in which  $\Omega_k = C_k P_{k/k-1} C_k^T + R_k$ ,  $P_{k/k} = P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1}$  and  $P_{k+1/k} = A_k P_{k/k} A_k^T + B_k Q_k B_k^T$ . It is common practice (see [1] – [5]) to linearise about the current conditional mean estimate, retain up to first order terms within the corresponding Taylor series and assume  $B_k = b_k(\hat{x}_{k/k})$ . This leads to

$$\begin{aligned} a_k(x) &\approx a_k(\hat{x}_{k/k}) + (x - \hat{x}_{k/k})^T \nabla a_k \Big|_{x=\hat{x}_{k/k}} \\ &= A_k x_k + \mu_k \end{aligned} \quad (12)$$

and

$$\begin{aligned} c_k(x_k) &\approx c_k(\hat{x}_{k/k-1}) + (x - \hat{x}_{k/k-1})^T \nabla c_k \Big|_{x=\hat{x}_{k/k-1}} \\ &= C_k x_k + \pi_k, \end{aligned} \quad (13)$$

where  $A_k = \nabla a_k(x) \Big|_{x=\hat{x}_{k/k}}$ ,  $C_k = \nabla c_k(x) \Big|_{x=\hat{x}_{k/k-1}}$ ,  $\mu_k = a_k(\hat{x}_{k/k}) - A_k \hat{x}_{k/k}$  and  $\pi_k = c_k(\hat{x}_{k/k-1}) - C_k \hat{x}_{k/k-1}$ . Substituting for  $\mu_k$  and  $\pi_k$  into (10) – (11) gives

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - c_k(\hat{x}_{k/k-1})), \quad (14)$$

$$\hat{x}_{k+1/k} = A_k(\hat{x}_{k/k}). \quad (15)$$

Note that nonlinearities enter into the state correction (14) and prediction (15), whereas linearised matrices  $A_k$ ,  $B_k$  and  $C_k$  are employed in the Riccati equation and gain calculations.

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“People take the longest possible paths, digress to numerous dead ends, and make all kinds of mistakes. Then historians come along and write summaries of this messy, nonlinear process and make it appear like a simple straight line.” *Dean L. Kamen*

In the case of scalar states, the linearisations are  $A_k = \left. \frac{\partial a_k}{\partial x} \right|_{x=\hat{x}_{k/k}}$  and  $C_k =$

$\left. \frac{\partial c_k}{\partial x} \right|_{x=\hat{x}_{k/k-1}}$ . In texts on optimal filtering, the recursions (14) – (15) are either called

a first-order EKF or simply an EKF, see [1] – [5]. Two higher order versions are developed below.

### 10.2.4 Second-Order Extended Kalman Filter

Truncating the series (1) after the second-order term and observing that  $(x - \hat{x}_{k/k})^T \nabla^T$  is a scalar yields

$$\begin{aligned} a_k(x) &\approx a_k(\hat{x}_{k/k}) + (x - \hat{x}_{k/k})^T \nabla a_k \Big|_{x=\hat{x}_{k/k}} + \frac{1}{2} (x - \hat{x}_{k/k})^T \nabla^T \nabla a_k \Big|_{x=\hat{x}_{k/k}} (x - \hat{x}_{k/k}), \\ &= a_k(\hat{x}_{k/k}) + (x - \hat{x}_{k/k})^T \nabla a_k \Big|_{x=\hat{x}_{k/k}} + \frac{1}{2} \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}} \\ &= A_k x_k + \mu_k, \end{aligned} \quad (16)$$

where  $A_k = \nabla a_k(x) \Big|_{x=\hat{x}_{k/k}}$  and  $\mu_k = a_k(\hat{x}_{k/k}) - A_k \hat{x}_{k/k} + \frac{1}{2} \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}}$ .

Similarly for the system output,

$$\begin{aligned} c_k(x) &\approx c_k(\hat{x}_{k/k-1}) + (x_k - \hat{x}_{k/k-1})^T \nabla c_k \Big|_{x=\hat{x}_{k/k-1}} \\ &\quad + \frac{1}{2} (x_k - \hat{x}_{k/k-1})^T \nabla^T \nabla c_k \Big|_{x=\hat{x}_{k/k-1}} (x_k - \hat{x}_{k/k-1}) \\ &= c_k(\hat{x}_{k/k-1}) + (x_k - \hat{x}_{k/k-1})^T \nabla c_k \Big|_{x=\hat{x}_{k/k-1}} + \frac{1}{2} \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}} \\ &= C_k x_k + \pi_k, \end{aligned} \quad (17)$$

where  $C_k = \nabla c_k(x) \Big|_{x=\hat{x}_{k/k-1}}$  and  $\pi_k = c_k(\hat{x}_{k/k-1}) - C_k \hat{x}_{k/k-1} + \frac{1}{2} \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}}$ .

Substituting for  $\mu_k$  and  $\pi_k$  into the filtering and prediction recursions (10) – (11) yields the second-order EKF

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“It might be a good idea if the various countries of the world would occasionally swap history books, just to see what other people are doing with the same set of facts.” *William E. Vaughan*

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k \left( z_k - c_k(\hat{x}_{k/k-1}) - \frac{1}{2} \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}} \right), \quad (18)$$

$$\hat{x}_{k+1/k} = a_k(\hat{x}_{k/k}) + \frac{1}{2} \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}}. \quad (19)$$

The above form is described in [2]. The further simplifications  $\nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}} \approx \text{tr} \left( \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}} \right)$  and  $\nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}} \approx \text{tr} \left( \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}} \right)$  are assumed in [4], [5].

### 10.2.5 Third-Order Extended Kalman Filter

Higher order EKF's can be realised just as elegantly as its predecessors. Retaining up to third-order terms within (1) results in

$$\begin{aligned} a_k(x) &\approx a_k(\hat{x}_{k/k}) + (x - \hat{x}_{k/k}) \nabla a_k \Big|_{x=\hat{x}_{k/k}} + \frac{1}{2} \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}} \\ &\quad + \frac{1}{6} \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}} (x_k - \hat{x}_{k/k}) \nabla a_k \Big|_{x=\hat{x}_{k/k}} \\ &= A_k x_k + \mu_k, \end{aligned} \quad (20)$$

where

$$A_k = \nabla a_k(x) \Big|_{x=\hat{x}_{k/k}} + \frac{1}{6} \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}} \quad (21)$$

and  $\mu_k = a_k(\hat{x}_{k/k}) - A_k \hat{x}_{k/k} + \frac{1}{2} \nabla P_{k/k} \nabla^T a_k \Big|_{x=\hat{x}_{k/k}}$ . Similarly, for the output nonlinearity it is assumed that

$$\begin{aligned} c_k(x_k) &\approx c_k(\hat{x}_{k/k-1}) + (x_k - \hat{x}_{k/k-1}) \nabla c_k \Big|_{x=\hat{x}_{k/k-1}} + \frac{1}{2} \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}} \\ &\quad + \frac{1}{6} \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}} (x_k - \hat{x}_{k/k-1}) \nabla c_k \Big|_{x=\hat{x}_{k/k-1}} \\ &= C_k x_k + \pi_k, \end{aligned} \quad (22)$$

where

$$C_k = \nabla c_k(x) \Big|_{x=\hat{x}_{k/k-1}} + \frac{1}{6} \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}} \quad (23)$$

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“Following the light of the sun, we left the old world.” *Christopher Columbus*

and  $\pi_k = c_k(\hat{x}_{k/k-1}) - C_k \hat{x}_{k/k-1} + \frac{1}{6} \nabla P_{k/k-1} \nabla^T c_k \Big|_{x=\hat{x}_{k/k-1}}$ . The resulting third-order EKF is defined by (18) – (19) in which the gain is now calculated using (21) and (23).

**Example 1.** Consider a linear state evolution  $x_{k+1} = Ax_k + w_k$ , with  $A = 0.5$ ,  $w_k \in \mathbb{R}$ ,  $Q = 0.05$ , a nonlinear output mapping  $y_k = \sin(x_k)$  and noisy observations  $z_k = y_k + v_k$ ,  $v_k \in \mathbb{R}$ . The first-order EKF for this problem is given by

$$\begin{aligned}\hat{x}_{k/k} &= \hat{x}_{k/k-1} + L_k(z_k - \sin(\hat{x}_{k/k-1})), \\ \hat{x}_{k+1/k} &= A\hat{x}_{k/k},\end{aligned}$$

where  $L_k = P_{k/k-1} C_k^T \Omega_k^{-1}$ ,  $\Omega_k = C_k P_{k/k-1} C_k^T + R_k$ ,  $C_k = \cos(\hat{x}_{k/k-1})$ ,  $P_{k/k} = P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1}$  and  $P_{k+1/k} = A P_{k/k} A^T + Q_k$ . The filtering step within the second-order EKF is amended to

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k(z_k - \sin(\hat{x}_{k/k-1}) + \sin(\hat{x}_{k/k-1}) P_{k/k-1} / 2).$$

The modified output linearisation for the third-order EKF is

$$C_k = \cos(\hat{x}_{k/k-1}) + \sin(\hat{x}_{k/k-1}) P_{k/k-1} / 6.$$

Simulations were conducted in which the signal-to-noise-ratio was varied from 20 dB to 40 dB for  $N = 200,000$  realisations of Gaussian noise sequences. The mean-square-errors exhibited by the first, second and third-order EKFs are plotted in Fig. 1. The figure demonstrates that including higher-order Taylor series terms within the filter can provide small performance improvements but the benefit diminishes with increasing measurement noise.

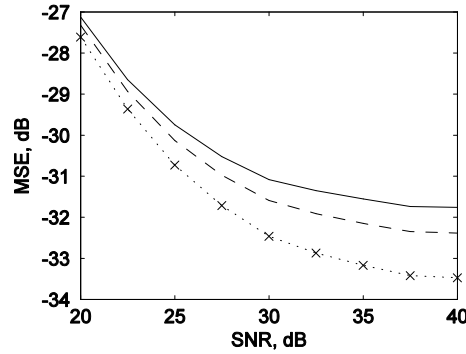


Fig. 1. Mean-square-error (MSE) versus signal-to-noise-ratio (SNR) for Example 1: first-order EKF (solid line), second-order EKF (dashed line) and third-order EKF (dotted-crossed line).

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“No two people see the external world in exactly the same way. To every separate person a thing is what he thinks it is – in other words, not a thing, but a think.” *Penelope Fitzgerald*

## 10.3 The Faux Algebraic Riccati Equation Technique

### 10.3.1 A Nonlinear Observer

The previously-described Extended-Kalman filters arise by linearising the signal model about the current state estimate and using the linear Kalman filter to predict the next estimate. This attempts to produce a locally optimal filter, however, it is not necessarily stable because the solutions of the underlying Riccati equations are not guaranteed to be positive definite. The faux algebraic Riccati technique [6] – [10] seeks to improve on EKF performance by trading off approximate optimality for stability. The familiar structure of the EKF is retained but stability is achieved by selecting a positive definite solution to a faux Riccati equation for the gain design.

Assume that data is generated by the following signal model comprising a stable, linear state evolution together with a nonlinear output mapping

$$x_{k+1} = Ax_k + Bw_k, \quad (24)$$

$$z_k = c_k(x_k) + v_k, \quad (25)$$

where the components of  $c_k(\cdot)$  are assumed to be continuous differentiable functions. Suppose that it is desired to calculate estimates of the states from the measurements. A nonlinear observer may be constructed having the form

$$\hat{x}_{k+1/k} = A\hat{x}_k + g_k(z_k - c(\hat{x}_{k/k-1})), \quad (26)$$

where  $g_k(\cdot)$  is a gain function to be designed. From (24) – (26), the state prediction error is given by

$$\tilde{x}_{k+1/k} = A\tilde{x}_{k/k-1} - g_k(\varepsilon_k) + w_k, \quad (27)$$

where  $\tilde{x}_k = x_k - \hat{x}_{k/k-1}$  and  $\varepsilon_k = z_k - c(\hat{x}_{k/k-1})$ . The Taylor series expansion of  $c_k(\cdot)$  to first order terms leads to  $\varepsilon_k \approx C_k \tilde{x}_{k/k-1} + v_k$ , where  $C_k = \nabla c_k(x)|_{x=\hat{x}_{k/k-1}}$ . The objective here is to design  $g_k(\varepsilon_k)$  to be a linear function of  $\tilde{x}_{k/k-1}$  to first order terms. It will be shown that for certain classes of problems, this objective can be achieved by a suitable choice of a linear bounded matrix function of the states  $D_k$ , resulting in the time-varying gain function  $g_k(\varepsilon_k) = K_k D_k \varepsilon_k$ , where  $K_k$  is a gain matrix of appropriate dimension. For example, consider  $x_k \in \mathbb{R}^n$  and  $z_k \in \mathbb{R}^m$ , which yield  $\varepsilon_k \in \mathbb{R}^m$  and  $C_k \in \mathbb{R}^{m \times n}$ . Suppose that a linearisation  $D_k \in \mathbb{R}^{p \times m}$  can be found so that  $\bar{C}_k = D_k C_k \in \mathbb{R}^{p \times m}$  possesses approximately constant terms. Then the locally linearised error (27) may be written as

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“The observer, when he seems to himself to be observing a stone, is really, if physics is to be believed, observing the effects of the stone upon himself.” *Bertrand Arthur William Russell*

$$\tilde{x}_{k+1/k} = (A - K_k \bar{C}_k) \tilde{x}_{k/k-1} - K_k D_k v_k + w_k. \quad (28)$$

If  $|\lambda_i(A)| < 1$ ,  $i = 1 \dots n$ , and if the pair  $(A, \bar{C}_k)$  is completely observable, then the asymptotic stability of (28) can be guaranteed by selecting the gain such that  $|\lambda_i(A - K_k \bar{C}_k)| < 1$ . A method for selecting the gain is described below.

### 10.3.2 Gain Selection

From (28), an approximate equation for the error covariance  $P_{k/k-1} = E\{\tilde{x}_{k/k-1} \tilde{x}_{k/k-1}^T\}$  is

$$P_{k+1/k} = (A - K_k \bar{C}_k) P_{k/k-1} (A - K_k \bar{C}_k)^T + K_k D_k R D_k^T K_k^T + Q, \quad (29)$$

which can be written as

$$P_{k/k} = P_{k/k-1} - P_{k/k-1} \bar{C}_k^T (\bar{C}_k P_{k/k-1} \bar{C}_k^T + D_k R D_k^T)^{-1} \bar{C}_k P_{k/k-1}, \quad (30)$$

$$P_{k+1/k} = A P_{k/k} A^T + Q. \quad (31)$$

In an EKF for the above problem, the gain is obtained by solving the above Riccati difference equation and calculating

$$K_k = P_{k/k-1} \bar{C}_k^T (\bar{C}_k P_{k/k-1} \bar{C}_k^T + D_k R D_k^T)^{-1}. \quad (32)$$

The faux algebraic Riccati equation approach [6] – [10] is motivated by connections between Riccati difference equation and algebraic Riccati equation solutions. Indeed, it is noted for some nonlinear problems that the gains can converge to a steady-state matrix [3]. This technique is also known as ‘covariance setting’. Following the approach of [10], the Riccati difference equation (30) may be masqueraded by the faux algebraic Riccati equation

$$\Sigma_k = \Sigma_k - \Sigma_k \bar{C}_k^T (\bar{C}_k \Sigma_k \bar{C}_k^T + D_k R D_k^T)^{-1} \bar{C}_k \Sigma_k. \quad (33)$$

That is, rather than solve (30), an arbitrary positive definite solution  $\Sigma_k$  is assumed instead and then the gain at each time  $k$  is calculated from (31) – (32) using  $\Sigma_k$  in place of  $P_{k/k-1}$ .

### 10.3.3 Tracking Multiple Signals

Consider the problem of tracking two frequency or phase modulated signals which may be modelled by equation (34), where  $a_k^{(1)}$ ,  $a_k^{(2)}$ ,  $\omega_k^{(1)}$ ,  $\omega_k^{(2)}$ ,  $\phi_k^{(1)}$ ,  $\phi_k^{(2)}$ ,  $\mu_a^{(1)}$ ,

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“The universe as we know it is a joint product of the observer and the observed.” *Pierre Teilhard De Chardin*

$\mu_a^{(2)}, \mu_\omega^{(1)}, \mu_\omega^{(2)} \in \mathbb{R}$  and  $w_k^{(1)}, \dots, w_k^{(6)} \in \mathbb{R}$  are zero-mean, uncorrelated, white processes with covariance  $Q = \text{diag}(\sigma_{w^{(1)}}^2, \dots, \sigma_{w^{(6)}}^2)$ . The states  $a_k^{(i)}, \omega_k^{(i)}$  and  $\phi_k^{(i)}$ ,  $i = 1, 2$ , represent the signals' instantaneous amplitude, frequency and phase components, respectively.

$$\begin{bmatrix} a_{k+1}^{(1)} \\ \omega_{k+1}^{(1)} \\ \phi_{k+1}^{(1)} \\ a_{k+1}^{(2)} \\ \omega_{k+1}^{(2)} \\ \phi_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} \mu_a^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_\omega^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_a^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_\omega^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_k^{(1)} \\ \omega_k^{(1)} \\ \phi_k^{(1)} \\ a_k^{(2)} \\ \omega_k^{(2)} \\ \phi_k^{(2)} \end{bmatrix} + \begin{bmatrix} w_k^{(1)} \\ w_k^{(2)} \\ w_k^{(3)} \\ w_k^{(4)} \\ w_k^{(5)} \\ w_k^{(6)} \end{bmatrix}. \quad (34)$$

Let

$$\begin{bmatrix} z_k^{(1)} \\ z_k^{(2)} \\ z_k^{(3)} \\ z_k^{(4)} \end{bmatrix} = \begin{bmatrix} a_k^{(1)} \cos \phi_k^{(1)} \\ a_k^{(1)} \sin \phi_k^{(1)} \\ a_k^{(2)} \cos \phi_k^{(2)} \\ a_k^{(2)} \sin \phi_k^{(2)} \end{bmatrix} + \begin{bmatrix} v_k^{(1)} \\ v_k^{(2)} \\ v_k^{(3)} \\ v_k^{(4)} \end{bmatrix} \quad (35)$$

denote the complex baseband observations, where  $v_k^{(1)}, \dots, v_k^{(4)} \in \mathbb{R}$  are zero-mean, uncorrelated, white processes with covariance  $R = \text{diag}(\sigma_{v^{(1)}}^2, \dots, \sigma_{v^{(4)}}^2)$ . Expanding the prediction error to linear terms yields  $C_k = [C_k^{(1)} \ C_k^{(2)}]$ , where

$$C_k^{(i)} = \begin{bmatrix} \cos \hat{\phi}_{k/k-1}^{(i)} & 0 & -\hat{a}_{k/k-1}^{(i)} \sin \hat{\phi}_{k/k-1}^{(i)} \\ \sin \hat{\phi}_{k/k-1}^{(i)} & 0 & \hat{a}_{k/k-1}^{(i)} \cos \hat{\phi}_{k/k-1}^{(i)} \end{bmatrix}.$$

This form suggests the choice  $D_k = \begin{bmatrix} D_k^{(1)} \\ D_k^{(2)} \end{bmatrix}$ , where

$$D_k^{(i)} = \begin{bmatrix} \cos \hat{\phi}_{k/k-1}^{(i)} & \sin \hat{\phi}_{k/k-1}^{(i)} \\ -\sin \hat{\phi}_{k/k-1}^{(i)} / \hat{a}_{k/k-1}^{(i)} & \cos \hat{\phi}_{k/k-1}^{(i)} / \hat{a}_{k/k-1}^{(i)} \end{bmatrix}.$$

In the multiple signal case, the linearisation  $\bar{C}_k = D_k C_k$  does not result in perfect decoupling. While the diagonal blocks reduce to  $\bar{C}_k^{(i,i)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , the off-diagonal blocks are

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“If you haven't found something strange during the day, it hasn't been much of a day.” *John Archibald Wheeler*

$$\bar{C}_k^{(i,j)} = \begin{bmatrix} \cos(\hat{\phi}_{k/k-1}^{(i)} - \hat{\phi}_{k/k-1}^{(j)}) & 0 & \hat{a}_{k/k-1}^{(j)} \sin(\hat{\phi}_{k/k-1}^{(i)} - \hat{\phi}_{k/k-1}^{(j)}) \\ -\frac{1}{\hat{a}_{k/k-1}^{(i)}} \cos(\hat{\phi}_{k/k-1}^{(i)} - \hat{\phi}_{k/k-1}^{(j)}) & 0 & \frac{\hat{a}_{k/k-1}^{(j)}}{\hat{a}_{k/k-1}^{(i)}} \cos(\hat{\phi}_{k/k-1}^{(i)} - \hat{\phi}_{k/k-1}^{(j)}) \end{bmatrix}.$$

Assuming a symmetric positive definite solution to (33) of the form  $\Sigma_k =$

$$\begin{bmatrix} \Sigma_k^a & 0 & 0 \\ 0 & \Sigma_k^\omega & \Sigma_k^{\omega\phi} \\ 0 & \Sigma_k^{\phi\omega} & \Sigma_k^\phi \end{bmatrix}, \text{ with } \Sigma_k^a, \Sigma_k^\omega, \Sigma_k^{\omega\phi}, \Sigma_k^\phi \in \mathbb{R} \text{ and choosing the gains according}$$

$$\text{to (32) yields } K_k = \begin{bmatrix} K_k^a & 0 \\ 0 & K_k^\omega \\ 0 & K_k^\phi \end{bmatrix}, \text{ where } K_k^a = \Sigma_k^a (\Sigma_k^a + \sigma_v^2)^{-1}, K_k^\omega = \Sigma_k^{\omega\phi} (\Sigma_k^\phi +$$

$\sigma_v^2 \hat{a}_{k/k}^{-2})^{-1}$  and  $K_k^\phi = \Sigma_k^\phi (\Sigma_k^\phi + \sigma_v^2 \hat{a}_{k/k}^{-2})^{-1}$ . The nonlinear observer then becomes

$$\begin{aligned} \hat{a}_{k/k}^{(i)} &= \hat{a}_{k/k-1}^{(i)} + \Sigma_k^a (z_k^{(1)} \cos \hat{\phi}_{k/k-1}^{(i)} + z_k^{(2)} \sin \hat{\phi}_{k/k-1}^{(i)}) (\Sigma_k^a + \sigma_v^2)^{-1}, \\ \hat{\omega}_{k/k}^{(i)} &= \hat{\omega}_{k/k-1}^{(i)} + \Sigma_k^{\omega\phi} (z_k^{(1)} \cos \hat{\phi}_{k/k-1}^{(i)} + z_k^{(2)} \sin \hat{\phi}_{k/k-1}^{(i)}) (\hat{a}_{k/k-1}^{(i)} \Sigma_k^\phi + \sigma_v^2 / a_{k/k-1}^{(i)})^{-1}, \\ \hat{\phi}_{k/k}^{(i)} &= \hat{\phi}_{k/k-1}^{(i)} + \Sigma_k^\phi (z_k^{(1)} \cos \hat{\phi}_{k/k-1}^{(i)} + z_k^{(2)} \sin \hat{\phi}_{k/k-1}^{(i)}) (\hat{a}_{k/k-1}^{(i)} \Sigma_k^\phi + \sigma_v^2 / a_{k/k-1}^{(i)})^{-1}. \end{aligned}$$

### 10.3.4 Stability Conditions

In order to establish conditions for the error system (28) to be asymptotically

stable, the problem is recast in a passivity framework as follows. Let  $w = \begin{bmatrix} w_k^{(1)} \\ w_k^{(2)} \\ \vdots \\ w_k^{(m)} \end{bmatrix}$ ,

$$e = \begin{bmatrix} e_k^{(1)} \\ e_k^{(2)} \\ \vdots \\ e_k^{(m)} \end{bmatrix} \in \mathbb{R}^m. \text{ Consider the configuration of Fig. 2, in which there is a}$$

cascade of a stable linear system  $\mathcal{G}$  and a nonlinear function matrix  $\gamma(\cdot)$  acting on  $e$ . It follows from the figure that

$$e = w - \mathcal{G}\gamma(e). \quad (36)$$

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“Discovery consists of seeing what everyone has seen and thinking what nobody has thought.” *Albert Szent-Görgyi*



Let  $\nabla$  denote a forward difference operator with  $\nabla e_k = e_k^{(i)} - e_{k-1}^{(i)}$ . It is assumed that  $\gamma(\cdot)$  satisfies some sector conditions which may be interpreted as bounds existing on the slope of the components of  $\gamma(\cdot)$ ; see Theorem 14, p. 7 of [11].

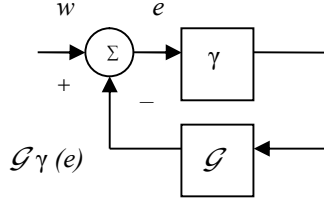


Fig. 2. Nonlinear error system configuration.

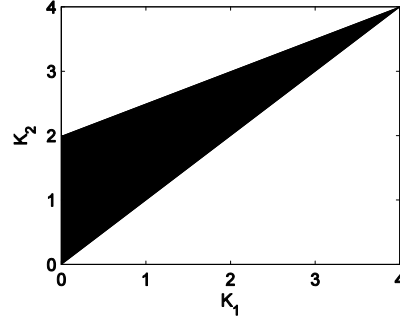


Fig. 3. Stable gain space for Example 2.

**Lemma 1 [10]:** Consider the system (36), where  $w, e \in \mathbb{R}^m$ . Suppose that  $\gamma(\cdot)$  consists of  $m$  identical, noninteracting nonlinearities, with  $\gamma(e_k^{(i)})$  monotonically increasing in the sector  $[0, \beta]$ ,  $\beta \geq 0$ ,  $\beta \in \mathbb{R}$ , that is,

$$0 \leq \gamma(e_k^{(i)}) / e_k^{(i)} \leq \beta \quad (37)$$

for all  $e_k^{(i)} \in \mathbb{R}$ ,  $e_k^{(i)} \neq 0$ . Assume that  $\mathcal{G}$  is a causal, stable, finite-gain, time-invariant map  $\mathbb{R}^m \rightarrow \mathbb{R}^m$ , having a  $z$ -transform  $G(z)$ , which is bounded on the unit circle. Let  $I$  denote an  $m \times m$  identity matrix. Suppose that for some  $q > 0$ ,  $q \in \mathbb{R}$ , there exists a  $\delta \in \mathbb{R}$ , such that

$$\langle (G(z) + q\nabla G(z) + I\beta^{-1})e, e \rangle \geq \delta \langle e, e \rangle \quad (38)$$

for all  $e_k^{(i)} \in \mathbb{R}$ . Under these conditions  $w \in \ell_2$  implies  $e, \gamma(e_k^{(i)}) \in \ell_2$ .

**Proof:** From (36),  $\nabla w = \nabla e + \nabla G(z)\gamma(e)$  and  $w + q\nabla w = (G(z) + q\nabla G(z) + I\beta^{-1})\gamma(e) + e - I\beta^{-1}\gamma(e) + e - I\beta^{-1}\gamma(e) + q\nabla e$ . Then

$$\begin{aligned} \langle w + q\nabla w, \gamma(e) \rangle &\geq \langle e - I\beta^{-1}\gamma(e), \gamma(e) \rangle + \langle q\nabla e, \gamma(e) \rangle \\ &\quad + \langle (G(z) + q\nabla G(z) + I\beta^{-1})\gamma(e), \gamma(e) \rangle. \end{aligned} \quad (39)$$

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“The intelligent man finds almost everything ridiculous, the sensible man hardly anything.” Johann Wolfgang von Goethe

Consider the first term on the right hand side of (39). Since the  $\gamma(e)$  consists of noninteracting nonlinearities,  $\langle \gamma(e), e \rangle = \sum_{i=1}^m \langle \gamma(e^{(i)}), e^{(i)} \rangle$  and  $\langle e - I\beta^{-1}\gamma(e), \gamma(e) \rangle = \sum_{i=1}^m \langle e^{(i)} - \gamma(e^{(i)})I\beta^{-1}, e^{(i)} \rangle \geq 0$ . Using the approach of [11] together with the sector conditions on the identical noninteracting nonlinearities (37), it can be shown that expanding out the second term of (39) yields  $\langle \nabla e, \gamma(e) \rangle \geq 0$ . Using  $\|\nabla w\|_2 \leq 2\|w\|_2$  (from p. 192 of [11]), the Schwartz inequality and the triangle inequality, it can be shown that

$$\langle w + q\nabla w, \gamma(e) \rangle \leq (1 + 2q)\|w\|_2. \quad (40)$$

It follows from (38) – (40) that  $\|\gamma(e)\|_2^2 \leq (1 + 2q)\delta^{-1}\|w\|_2$ ; hence  $\gamma(e_k^{(i)}) \in \ell_2$ . Since the gain of  $G(z)$  is finite, it also follows that  $G(z)\gamma(e_k^{(i)}) \in \ell_2$ .

If  $G(z)$  is stable and bounded on the unit circle, then the test condition (38) becomes

$$\lambda_{\min} \{I + q(I - z^{-1}I)(G(z) + G^H(z)) + \beta^{-1}\} \geq \delta, \quad (41)$$

see pp. 175 and 194 of [11].

### 10.3.5 Applications

**Example 2 [10].** Consider a unity-amplitude frequency modulated (FM) signal modelled as  $\omega_{k+1} = \mu_\omega \omega_k + w_k$ ,  $\phi_{k+1} = \phi_k + \omega_k$ ,  $z_k^{(1)} = \cos(\phi_k) + v_k^{(1)}$  and  $z_k^{(2)} = \sin(\phi_k) + v_k^{(2)}$ . The error system for an FM demodulator may be written as

$$\begin{bmatrix} \tilde{\omega}_{k+1} \\ \tilde{\phi}_{k+1} \end{bmatrix} = \begin{bmatrix} \mu_\omega & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\omega}_k \\ \tilde{\phi}_k \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \sin(\tilde{\phi}_k) + w_k \quad (42)$$

for gains  $K_1, K_2 \in \mathbb{R}$  to be designed. In view of the form (36), the above error system is reformatted as

$$\begin{bmatrix} \tilde{\omega}_{k+1} \\ \tilde{\phi}_{k+1} \end{bmatrix} = \begin{bmatrix} \mu_\omega & -K_1 \\ 1 & 1 - K_2 \end{bmatrix} \begin{bmatrix} \tilde{\omega}_k \\ \tilde{\phi}_k \end{bmatrix} + \gamma \left( \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\omega}_k \\ \tilde{\phi}_k \end{bmatrix} \right) + w_k, \quad (43)$$

where  $\gamma(x) = x - \sin(x)$ . The z-transform of the linear part of (43) is  $G(z) = (K_2 z + K_2 + \mu_\omega K_1) (z^2 + (K_2 - 1 - \mu_\omega)z + K_1 + 1 - \mu_\omega K_2)^{-1}$ . The nonlinearity satisfies the sector condition (37) for  $\beta = 1.22$ . Candidate gains may be assessed by checking

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“He that does not offend cannot be honest.” *Thomas Paine*

that  $G(z)$  is stable and the test condition (41). The stable gain space calculated for the case of  $\mu_\omega = 0.9$  is plotted in Fig. 3. The gains are required to lie within the shaded region of the plot for the error system (42) to be asymptotically stable.

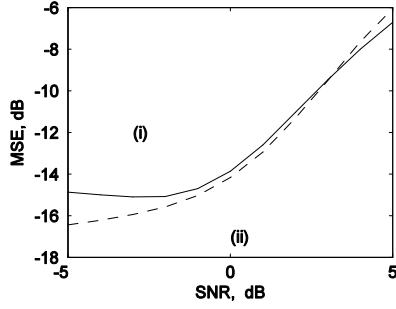


Fig. 4. Demodulation performance for Example 2: (i) EKF and (ii) Nonlinear observer.

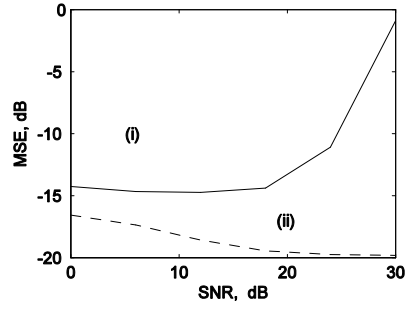


Fig. 5. Demodulation performance for Example 3: (i) EKF and (ii) Nonlinear observer.

A speech utterance, namely, the phrase “Matlab is number one”, was sampled at 8 kHz and used to synthesize a unity-amplitude FM signal. An EKF demodulator was constructed for the above model with  $\sigma_w^2 = 0.02$ . In a nonlinear observer

design it was found that suitable parameter choices were  $\Sigma_k = \begin{bmatrix} 0.001 & 0.08 \\ 0.08 & 0.7 \end{bmatrix}$ . The

nonlinear observer gains were censored at each time  $k$  according to the stable gain space of Fig. 3. The results of a simulation study using 100 realisations of Gaussian measurement noise sequences are shown in Fig. 4. The figure demonstrates that enforcing stability can be beneficial at low SNR, at the cost of degraded high-SNR performance.

**Example 3 [10].** Suppose that there are two superimposed FM signals present in the same frequency channel. Neglecting observation noise, a suitable approximation of the demodulator error system in the form (36) is given by

$$\begin{bmatrix} \tilde{\omega}_{k+1}^{(1)} \\ \tilde{\phi}_{k+1}^{(1)} \\ \tilde{\omega}_{k+1}^{(2)} \\ \tilde{\phi}_{k+1}^{(2)} \end{bmatrix} = (A - K_k \bar{C}) \begin{bmatrix} \tilde{\omega}_k^{(1)} \\ \tilde{\phi}_k^{(1)} \\ \tilde{\omega}_k^{(2)} \\ \tilde{\phi}_k^{(2)} \end{bmatrix} - K_k \begin{bmatrix} \sin(\tilde{\phi}_k^{(1)}) - \tilde{\phi}_k^{(1)} \\ \sin(\tilde{\phi}_k^{(2)}) - \tilde{\phi}_k^{(2)} \end{bmatrix}, \quad (44)$$

where  $A = \text{diag}(A^{(1)}, A^{(1)})$ ,  $A^{(1)} = \begin{bmatrix} \mu_\omega & 0 \\ 1 & 1 \end{bmatrix}$ ,  $\bar{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . The linear part of (44) may be written as  $G(z) = \bar{C}(zI - (A - K_k \bar{C}))^{-1} K_k$ . Two 8-kHz speech

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“To avoid criticism, do nothing, say nothing, be nothing.” *Elbert Hubbard*

utterances, “Matlab is number one” and “Number one is Matlab”, centred at  $\pm 0.25$  rad/s, were used to synthesize two superimposed unity-amplitude FM signals. Simulations were conducted using 100 realisations of Gaussian measurement noise sequences. The test condition (41) was evaluated at each time  $k$  for the above parameter values with  $\beta = 1.2$ ,  $q = 0.001$ ,  $\delta = 0.82$  and used to censor the gains. The resulting co-channel demodulation performance is shown in Fig. 5. It can be seen that the nonlinear observer significantly outperforms the EKF at high SNR.

Two mechanisms have been observed for occurrence of outliers or faults within the co-channel demodulators. Firstly errors can occur in the state attribution, that is, there is correct tracking of some component speech message segments but the tracks are inconsistently associated with the individual signals. This is illustrated by the example frequency estimate tracks shown in Figs. 6 and 7. The solid and dashed lines in the figures indicate two sample co-channel frequency tracks. Secondly, the phase unwrapping can be erroneous so that the frequency tracks bear no resemblance to the underlying messages. These faults can occur without any significant deterioration in the error residual.

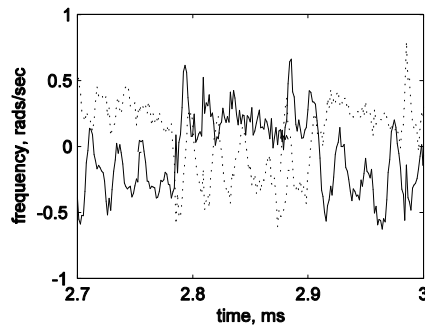


Fig. 6. Sample EKF frequency tracks for Example 3.

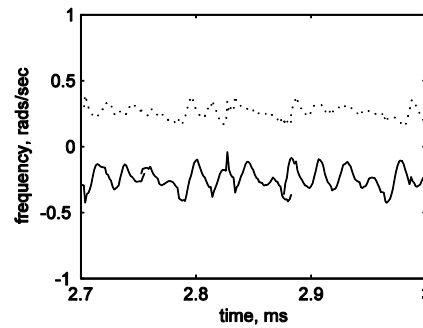


Fig. 7. Sample Nonlinear observer frequency tracks for Example 3.

The EKF demodulator is observed to be increasingly fault prone at higher SNR. This arises because lower SNR designs possess narrower bandwidths and so are less sensitive to nearby frequency components. The figures also illustrate the trade-off between stability and optimality. In particular, it can be seen from Fig. 6, that the sample EKF speech estimates exhibit faults in the state attribution. This contrasts with Fig. 7, where the nonlinear observer’s estimates exhibit stable state attribution at the cost of degraded speech fidelity.

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“You have enemies? Good. That means you’ve stood up for something, sometime in your life.”  
Winston Churchill

## 10.4 Robust Extended Kalman Filtering

### 10.4.1 Nonlinear Problem Statement

Consider again the nonlinear, discrete-time signal model (2), (7). It is shown below that the  $H_\infty$  techniques of Chapter 9 can be used to recast nonlinear filtering problems into a model uncertainty setting. The following discussion attends to state estimation, that is,  $C_{1,k} = I$  is assumed within the problem and solution presented in Section 9.3.2.

The Taylor series expansions of the nonlinear functions  $a_k(\cdot)$ ,  $b_k(\cdot)$  and  $c_k(\cdot)$  about filtered and predicted estimates  $\hat{x}_{k/k}$  and  $\hat{x}_{k/k-1}$  may be written as

$$a_k(x_k) = a_k(\hat{x}_{k/k}) + \nabla a_k(\hat{x}_{k/k})(x_k - \hat{x}_{k/k}) + \Delta_1(\tilde{x}_{k/k}), \quad (45)$$

$$b_k(x_k) = b_k(\hat{x}_{k/k}) + \Delta_2(\tilde{x}_{k/k}), \quad (46)$$

$$c_k(x_k) = c_k(\hat{x}_{k/k-1}) + \nabla c_k(\hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1}) + \Delta_3(\tilde{x}_{k/k-1}), \quad (47)$$

where  $\Delta_1(\cdot)$ ,  $\Delta_2(\cdot)$ ,  $\Delta_3(\cdot)$  are uncertainties that account for the higher order terms,  $\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}$  and  $\tilde{x}_{k/k-1} = x_k - \hat{x}_{k/k-1}$ . It is assumed that  $\Delta_1(\cdot)$ ,  $\Delta_2(\cdot)$  and  $\Delta_3(\cdot)$  are continuous operators mapping  $\ell_2 \rightarrow \ell_2$ , with  $H_\infty$  norms bounded by  $\delta_1$ ,  $\delta_2$  and  $\delta_3$ , respectively.

Substituting (45) – (47) into the nonlinear system (2), (7) gives the linearised system

$$x_{k+1} = A_k x_k + B_k w_k + \mu_k + \Delta_1(\tilde{x}_{k/k}) + \Delta_2(\tilde{x}_{k/k}) w_k, \quad (48)$$

$$z_k = C_k x_k + \pi_k + \Delta_3(\tilde{x}_{k/k-1}) + v_k, \quad (49)$$

where  $A_k = \nabla a_k(x)|_{x=\hat{x}_{k/k}}$ ,  $C_k = \nabla c_k(x)|_{x=\hat{x}_{k/k-1}}$ ,  $\mu_k = a_k(\hat{x}_{k/k}) - A_k \hat{x}_{k/k}$  and  $\pi_k = c_k(\hat{x}_{k/k-1}) - C_k \hat{x}_{k/k-1}$ .

Note that the first-order EKF for the above system arises by setting the uncertainties  $\Delta_1(\cdot)$ ,  $\Delta_2(\cdot)$  and  $\Delta_3(\cdot)$  to zero as

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k(z_k - c_k(\hat{x}_{k/k-1})), \quad (50)$$

$$\hat{x}_{k+1/k} = a_k(\hat{x}_{k/k}), \quad (51)$$

$$L_k = P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}, \quad (52)$$

$$P_{k/k} = P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} P_{k/k-1} C_k, \quad (53)$$

$$P_{k+1/k} = A_k P_{k/k-1} A_k^T + B_k Q_k B_k^T. \quad (54)$$

---

“Fight the good fight.” *Timothy 4:7*

## 10.4.2 Robust Solution

Following the approach in Section 9.2.4, instead of addressing the problem (48) – (49) which possesses uncertainties, an auxiliary  $H_\infty$  problem is defined as

$$x_{k+1} = A_k x_k + B_k w_k + \mu_k + s_k, \quad (55)$$

$$z_k = C_k x_k + \pi_k + v_k + t_k, \quad (56)$$

$$\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}, \quad (57)$$

where  $s_k = \Delta_1(\tilde{x}_{k/k}) + \Delta_2(\tilde{x}_{k/k})w_k$  and  $t_k = \Delta_3\tilde{x}_{k/k} \approx \Delta_3\tilde{x}_{k/k-1}$  are additional exogenous inputs satisfying

$$\|s_k\|_2^2 \leq \delta_1^2 \|\tilde{x}_{k/k}\|_2^2 + \delta_2^2 \|w_k\|_2^2, \quad (58)$$

$$\|t_k\|_2^2 \leq \delta_3^2 \|\tilde{x}_{k/k}\|_2^2 \leq \delta_3^2 \|\tilde{x}_{k/k-1}\|_2^2. \quad (59)$$

A sufficient solution to the auxiliary  $H_\infty$  problem (55) – (57) can be obtained by solving another problem in which  $w_k$  and  $v_k$  are scaled in lieu of the additional inputs  $s_k$  and  $t_k$ . The scaled  $H_\infty$  problem is defined by

$$x_{k+1} = A_k x_k + B_k c_w w_k + \mu_k, \quad (60)$$

$$z_k = C_k x_k + c_v v_k + \pi_k, \quad (61)$$

$$\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}, \quad (62)$$

where  $c_w, c_v \in \mathbb{R}$  are to be found.

**Lemma 2 [12]:** *The solution of the  $H_\infty$  problem (60) – (62), where  $v_k$  is scaled by*

$$c_v^2 = 1 - \gamma^2 \delta_1^2 - \gamma^2 \delta_3^2, \quad (63)$$

*and  $w_k$  is scaled by*

$$c_w^2 = c_v^2 (1 + \delta_2^2)^{-1}, \quad (64)$$

*is sufficient for the solution of the auxiliary  $H_\infty$  problem (55) – (57).*

**Proof:** *If the  $H_\infty$  problem (50) – (52) has been solved then there exists a  $\gamma \neq 0$  such that*

$$\begin{aligned} \|\tilde{x}_{k/k}\|_2^2 &\leq \gamma^2 (\|w_k\|_2^2 + \|s_k\|_2^2 + \|t_k\|_2^2 + \|v_k\|_2^2) \\ &\leq \gamma^2 (\|w_k\|_2^2 + \delta_1^2 \|\tilde{x}_{k/k}\|_2^2 + \delta_2^2 \|w_k\|_2^2 + \delta_3^2 \|\tilde{x}_{k/k}\|_2^2 + \|v_k\|_2^2), \end{aligned}$$

*which implies*

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“You can’t wait for inspiration. You have to go after it with a club.” *Jack London*

$$(1 - \gamma^2 \delta_1^2 - \gamma^2 \delta_3^2) \|\tilde{x}_{k/k}\|_2^2 \leq \gamma^2 ((1 + \delta_2^2) \|w_k\|_2^2 + \|v_k\|_2^2)$$

and

$$\|\tilde{x}_{k/k}\|_2^2 \leq \gamma^2 (c_w^{-2} \|w_k\|_2^2 + c_v^{-2} \|v_k\|_2^2).$$

The robust first-order extended Kalman filter for state estimation is given by (50) – (52),

$$P_{k/k} = P_{k/k-1} - P_{k/k-1} \begin{bmatrix} I & C_k^T \end{bmatrix} \begin{bmatrix} P_{k/k-1} - \gamma^2 I & P_{k/k-1} C_k^T \\ C_k P_{k/k-1} & R_k + C_k P_{k/k-1} C_k^T \end{bmatrix}^{-1} \begin{bmatrix} I \\ C_k \end{bmatrix} P_{k/k-1}$$

and (54). As discussed in Chapter 9, a search is required for a minimum  $\gamma$  such

that  $\begin{bmatrix} P_{k/k-1} - \gamma^2 I & P_{k/k-1} C_k^T \\ C_k P_{k/k-1} & R_k + C_k P_{k/k-1} C_k^T \end{bmatrix} > 0$  and  $P_{k/k-1} > 0$  over  $k \in [1, N]$ . An illustration is provided below.

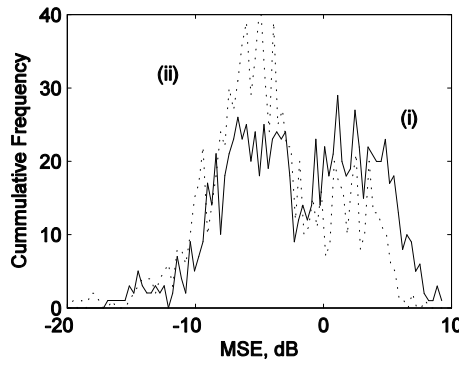


Fig. 8. Histogram of demodulator mean-square-error for Example 4: (i) first-order EKF (solid line) and first-order robust EKF (dotted line).

**Example 4 [12].** Suppose that an FM signal is generated by

$$\omega_{k+1} = \mu_\omega \omega_k + w_k, \quad (65)$$

$$\phi_{k+1} = \arctan(\mu_\phi \phi_k + \omega_k), \quad (66)$$

$$z_k^{(1)} = \cos(\phi_k) + v_k^{(1)}, \quad (67)$$

$$z_k^{(2)} = \sin(\phi_k) + v_k^{(2)}. \quad (68)$$

The objective is to construct an FM demodulator that produces estimates of the frequency message  $\omega_k$  from the noisy in-phase and quadrature measurements  $z_k^{(1)}$  and  $z_k^{(2)}$ , respectively. Simulations were conducted with  $\mu_\omega = 0.9$ ,  $\mu_\phi = 0.99$  and

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“Most pioneers are at the mercy of doubt at the beginning, whether of their worth, of their theories, or of the whole enigmatic field in which they labour.” *Johann Wolfgang von Goethe*

$\sigma_{v^{(1)}}^2 = \sigma_{v^{(2)}}^2 = 0.001$ . It was found for  $\sigma_w^2 < 0.1$ , where the state behaviour is almost linear, a robust EKF does not improve on the EKF. However, when  $\sigma_w^2 = 1$ , the problem is substantially nonlinear and a performance benefit can be observed. A robust EKF demodulator was designed with

$$x_k = \begin{bmatrix} \phi_k \\ \omega_k \end{bmatrix}, A_k = \begin{bmatrix} \frac{\mu_\phi}{(\mu_\phi \hat{\phi}_{k/k} + \hat{\omega}_{k/k})^2 + 1} & \frac{\mu_\phi}{(\mu_\phi \hat{\phi}_{k/k} + \hat{\omega}_{k/k})^2 + 1} \\ 0 & \mu_\omega \end{bmatrix},$$

$$C_k = \begin{bmatrix} -\sin(\hat{\phi}_{k/k-1}) & 0 \\ \cos(\hat{\phi}_{k/k-1}) & 0 \end{bmatrix},$$

$\delta_1 = 0.1$ ,  $\delta_2 = 4.5$  and  $\delta_3 = 0.001$ . It was found that  $\gamma = 1.38$  was sufficient for  $P_{k/k-1}$  of the above Riccati difference equation to always be positive definite. A histogram of the observed frequency estimation error is shown in Fig. 8, which demonstrates that the robust demodulator provides improved mean-square-error performance. For sufficiently large  $\sigma_w^2$ , the output of the above model will resemble a digital signal, in which case a detector may outperform a demodulator.

## 10.5 Nonlinear Smoothing

### 10.5.1 Approximate Minimum-Variance Smoother

Consider again a nonlinear estimation problem where  $x_{k+1} = a_k(x_k) + B_k w_k$ ,  $z_k = c_k(x_k) + v_k$ , with  $x_k \in \mathbb{R}$ , in which the nonlinearities  $a_k(\cdot)$ ,  $c_k(\cdot)$  are assumed to be smooth, differentiable functions of appropriate dimension. The linearisations akin to Extended Kalman filtering may be applied within the smoothers described in Chapter 7 in the pursuit of performance improvement. The fixed-lag, Fraser-Potter and Rauch-Tung-Striebel smoother recursions are easier to apply as they are less complex. The application of the minimum-variance smoother can yield approximately optimal estimates when the problem becomes linear, provided that the underlying assumptions are correct.

**Procedure 1.** An approximate minimum-variance smoother for output estimation can be implemented via the following three-step procedure.

Step 1. Operate

$$\alpha_k = -\Omega_k^{1/2} (z_k - c_k(\hat{x}_{k/k-1})), \quad (69)$$

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k (z_k - c_k(\hat{x}_{k/k-1})), \quad (70)$$

$$\hat{x}_{k+1/k} = a_k(\hat{x}_{k/k}), \quad (71)$$

---

“You can recognize a pioneer by the arrows in his back” *Beverly Rubik*



on the measurement  $z_k$ , where  $L_k = P_{k/k-1} C_k^T \Omega_k^{-1}$ ,

$$\begin{aligned} \Omega_k &= C_k P_{k/k-1} C_k^T + R_k, \\ P_{k/k} &= P_{k/k-1} - P_{k/k-1} C_k^T \Omega_k^{-1} C_k P_{k/k-1}, \\ P_{k+1/k} &= A_k P_{k/k} A_k^T + B_k Q_k B_k^T, \\ A_k &= \left. \frac{\partial a_k}{\partial x} \right|_{x=\hat{x}_{k/k}} \quad \text{and} \quad C_k = \left. \frac{\partial c_k}{\partial x} \right|_{x=\hat{x}_{k/k-1}}. \end{aligned} \quad (72)$$

Step 2. Operate (69) – (71) on the time-reversed transpose of  $\alpha_k$ . Then take the time-reversed transpose of the result to obtain  $\beta_k$ .

Step 3. Calculate the smoothed output estimate from

$$\hat{y}_{k/N} = z_k - R_k \beta_k. \quad (73)$$

### 10.5.2 Robust Smoother

From the arguments within Chapter 9, a smoother that is robust to uncertain  $w_k$  and  $v_k$  can be realised by replacing the error covariance correction (72) by

$$P_{k/k} = P_{k/k-1} - P_{k/k-1} \begin{bmatrix} C_k^T & C_k^T \end{bmatrix} \begin{bmatrix} C_k P_{k/k-1} C_k^T - \gamma^2 I & C_k P_{k/k-1} C_k^T \\ C_k P_{k/k-1} C_k^T & R_k + C_k P_{k/k-1} C_k^T \end{bmatrix}^{-1} \begin{bmatrix} C_k \\ C_k \end{bmatrix} P_{k/k-1}$$

within Procedure 1. As discussed in Chapter 9, a search for a minimum  $\gamma$  such that

$$\begin{bmatrix} C_k P_{k/k-1} C_k^T - \gamma^2 I & C_k P_{k/k-1} C_k^T \\ C_k P_{k/k-1} C_k^T & R_k + C_k P_{k/k-1} C_k^T \end{bmatrix} > 0 \text{ and } P_{k/k-1} > 0 \text{ over } k \in [1, N] \text{ is desired.}$$

### 10.5.3 Application

Returning to the problem of demodulating a unity-amplitude FM signal, let  $x_k =$

$$\begin{bmatrix} \omega_k \\ \phi_k \end{bmatrix}, \quad A = \begin{bmatrix} \mu_w & 0 \\ 1 & \mu_\phi \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad z_k^{(1)} = \cos(\phi_k) + v_k^{(1)}, \quad z_k^{(2)} = \sin(\phi_k) + v_k^{(2)},$$

where  $\omega_k$ ,  $\phi_k$ ,  $z_k$  and  $v_k$  denote the instantaneous frequency message, instantaneous phase, complex observations and measurement noise respectively. A zero-mean voiced speech utterance “a e i o u” was sampled at 8 kHz, for which estimates  $\hat{\mu}_w = 0.97$  and  $\hat{\sigma}_w^2 = 0.053$  were obtained using an expectation maximisation algorithm. An FM discriminator output [13],

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“The farther the experiment is from theory, the closer it is to the Nobel Prize.” *Irène Joliot-Curie, winner of the 1935 Nobel Prize in Chemistry*

$$z_k^{(3)} = \left( z_k^{(1)} \frac{dz_k^{(2)}}{dt} - z_k^{(2)} \frac{dz_k^{(1)}}{dt} \right) \left( (z_k^{(1)})^2 + (z_k^{(2)})^2 \right)^{-1}, \quad (74)$$

serves as a benchmark and as an auxiliary frequency measurement for the above

smoother. The innovations within Steps 1 and 2 are given by  $\begin{bmatrix} z_k^{(1)} \\ z_k^{(2)} \\ z_k^{(3)} \end{bmatrix} - \begin{bmatrix} \cos(\hat{x}_k^{(2)}) \\ \sin(\hat{x}_k^{(2)}) \\ \hat{x}_k^{(1)} \end{bmatrix}$

and  $\begin{bmatrix} \alpha_k^{(1)} \\ \alpha_k^{(2)} \\ \alpha_k^{(3)} \end{bmatrix} - \begin{bmatrix} \cos(\hat{x}_k^{(2)}) \\ \sin(\hat{x}_k^{(2)}) \\ \hat{x}_k^{(1)} \end{bmatrix}$  respectively. A unity-amplitude FM signal was

synthesized using  $\mu_\phi = 0.99$  and the SNR was varied in 1.5 dB steps from 3 dB to 15 dB. The mean-square errors were calculated over 200 realisations of Gaussian measurement noise and are shown in Fig. 9. It can be seen from the figure, that at 7.5 dB SNR, the first-order EKF improves on the FM discriminator MSE by about 12 dB. The improvement arises because the EKF demodulator exploits the signal model whereas the FM discriminator does not. The figure shows that the approximate minimum-variance smoother further reduces the MSE by about 2 dB, which illustrates the advantage of exploiting all the data in the time interval. In the robust designs, searches for minimum values of  $\gamma$  were conducted such that the corresponding Riccati difference equation solutions were positive definite over each noise realisation. It can be seen from the figure at 7.5 dB SNR that the robust EKF provides about a 1 dB performance improvement compared to the EKF, whereas the approximate minimum-variance smoother and the robust smoother performance are indistinguishable.

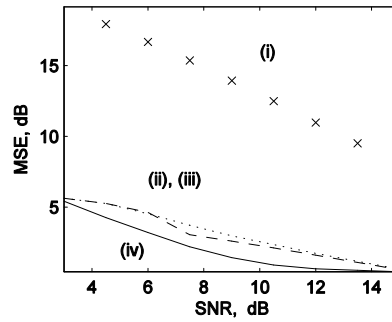


Fig. 9. FM demodulation performance comparison: (i) FM discriminator (crosses), (ii) first-order EKF (dotted line), (iii) Robust EKF (dashed line), (iv) approximate minimum-variance smoother and robust smoother (solid line).

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“They thought I was crazy, absolutely mad.” *Barbara McClintock, winner of the 1983 Nobel Prize in Physiology or Medicine*

This nonlinear example illustrates once again that smoothers can outperform filters. Since a first-order speech model is used and the Taylor series are truncated after the first-order terms, some model uncertainty is present, and so the robust designs demonstrate a marginal improvement over the EKF.

## 10.6 Constrained Filtering and Smoothing

### 10.6.1 Background

Constraints often appear within navigation problems. For example, vehicle trajectories are typically constrained by road, tunnel and bridge boundaries. Similarly, indoor pedestrian trajectories are constrained by walls and doors. However, as constraints are not easily described within state-space frameworks, many techniques for constrained filtering and smoothing are reported in the literature. An early technique for constrained filtering involves augmenting the measurement vector with perfect observations [14]. The application of the perfect-measurement approach to filtering and fixed-interval smoothing is described in [15].

Constraints can be applied to state estimates, see [16], where a positivity constraint is used within a Kalman filter and a fixed-lag smoother. Three different state equality constraint approaches, namely, maximum-probability, mean-square and projection methods are described in [17]. Under prescribed conditions, the perfect-measurement and projection approaches are equivalent [5], [18], which is identical to applying linear constraints within a form of recursive least squares.

In the state equality constrained methods [5], [16] – [18], a constrained estimate can be calculated from a Kalman filter’s unconstrained estimate at each time step. Constraint information could also be embedded within nonlinear models for use with EKFs. A simpler, low-computation-cost technique that avoids EKF stability problems and suits real-time implementation is described in [19]. In particular, an on-line procedure is proposed that involves using nonlinear functions to censor the measurements and subsequently applying the minimum-variance filter recursions. An off-line procedure for retrospective analyses is also described, where the minimum-variance fixed-interval smoother recursions are applied to the censored measurements. In contrast to the afore-mentioned techniques, which employ constraint matrices and vectors, here constraint information is represented by an exogenous input process. This approach uses the Bounded Real Lemma which enables the nonlinearities to be designed so that the filtered and smoothed estimates satisfy a performance criterion.

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“If at first, the idea is not absurd, then there is no hope for it.” *Albert Einstein*

### 10.3.2 Problem Statement

The ensuing discussion concerns odd and even functions which are defined as follows. A function  $g_o$  of  $X$  is said to be odd if  $g_o(-X) = -g_o(X)$ . A function  $f_e$  of  $X$  is said to be even if  $f_e(-X) = f_e(X)$ . The product of  $g_o$  and  $f_e$  is an odd function since  $g_o(-X)f_e(-X) = -g_o(X)f_e(X)$ .

Problems are considered where stochastic random variables are subjected to inequality constraints. Therefore, nonlinear censoring functions are introduced whose outputs are constrained to lie within prescribed bounds. Let  $\beta \in \mathbb{R}^p$  and  $g_o : \mathbb{R}^p \rightarrow \mathbb{R}^p$  denote a constraint vector and an odd function of a random variable  $X \in \mathbb{R}^p$  about its expected value  $E\{X\}$ , respectively. Define the censoring function

$$g(X) = E\{X\} + g_o(X, \beta), \quad (75)$$

where

$$g_o(X, \beta) = \begin{cases} \beta & \text{if } \beta \leq X - E\{X\} \\ X - E\{X\} & \text{if } -\beta < X - E\{X\} < \beta \\ -\beta & \text{if } X - E\{X\} \leq -\beta \end{cases} \quad (76)$$

By inspection of (75) – (76),  $g(X)$  is constrained within  $E\{X\} \pm \beta$ . Suppose that the probability density function of  $X$  about  $E\{X\}$  is even, that is, is symmetric about  $E\{X\}$ . Under these conditions, the expected value of  $g(X)$  is given by

$$\begin{aligned} E\{g(X)\} &= \int_{-\infty}^{\infty} g(x) f_e(x) dx \\ &= E\{X\} \int_{-\infty}^{\infty} f_e(x) dx + \int_{-\infty}^{\infty} g_o(x, \beta) f_e(x) dx \\ &= E\{X\}. \end{aligned} \quad (77)$$

since  $\int_{-\infty}^{\infty} f_e(x) dx = 1$  and the product  $g_o(x, \beta) f_e(x)$  is odd.

Thus, a constraining process can be modelled by a nonlinear function. Equation (77) states that  $g(X)$  is unbiased, provided that  $g_o(X, \beta)$  and  $f_e(X)$  are odd and even functions about  $E\{X\}$ , respectively. In the analysis and examples that follow, attention is confined to systems having zero-mean inputs, states and outputs, in which case the censoring functions are also centred on zero, that is,  $E\{X\} = 0$ .

---

“It was not easy for a person brought up in the ways of classical thermodynamics to come around to the idea that gain of entropy eventually is nothing more nor less than loss of information.” *Gilbert Newton Lewis*

Let  $w_k = [w_{1,k} \ \dots \ w_{m,k}]^T \in \mathbb{R}^m$  represent a stochastic white input process having an even probability density function, with  $E\{w_k\} = 0$ ,  $E\{w_j w_k^T\} = Q_k \delta_{jk}$ , in which  $\delta_{jk}$  denotes the Kronecker delta function. Suppose that the states of a system  $\mathcal{G}: \mathbb{R}^m \rightarrow \mathbb{R}^p$  are realised by

$$x_{k+1} = A_k x_k + B_k w_k, \quad (78)$$

where  $A_k \in \mathbb{R}^{n \times n}$  and  $B_k \in \mathbb{R}^{n \times m}$ . Since  $w_k$  is zero-mean, it follows that linear combinations of the states are also zero-mean. Suppose also that the system outputs,  $y_k$ , are generated by

$$y_k = \begin{bmatrix} y_{1,k} \\ \vdots \\ y_{p,k} \end{bmatrix} = \begin{bmatrix} g_o(C_{1,k} x_k, \theta_{1,k}) \\ \vdots \\ g_o(C_{p,k} x_k, \theta_{p,k}) \end{bmatrix}, \quad (79)$$

where  $C_{j,k}$  is the  $j^{\text{th}}$  row of  $C_k \in \mathbb{R}^{p \times m}$ ,  $\theta_k = [\theta_{1,k} \ \dots \ \theta_{p,k}]^T \in \mathbb{R}^p$  is an input constraint process and  $g_o(C_{j,k} x_k, \theta_{j,k})$ ,  $j = 1, \dots, p$ , is an odd censoring function centred on zero. The outputs  $y_{j,k}$  are constrained to lie within  $\pm \theta_{j,k}$ , that is,

$$-\theta_{j,k} \leq y_{j,k} \leq \theta_{j,k}. \quad (80)$$

For example, if the system outputs represent the trajectories of pedestrians within a building then the constraint process could include knowledge about wall, floor and ceiling positions. Similarly, a vehicle trajectory constraint process could include information about building and road boundaries.

Assume that observations  $z_k = y_k + v_k$  are available, where  $v_k \in \mathbb{R}^p$  is a stochastic, white measurement noise process having an even probability density function, with  $E\{v_k\} = 0$ ,  $E\{v_k v_k^T\} = R_k \delta_{jk}$  and  $E\{w_j v_k^T\} = 0$ . It is convenient to define the stacked vectors  $y = [y_1^T \ \dots \ y_N^T]^T$  and  $\theta = [\theta_1^T \ \dots \ \theta_N^T]^T$ . It follows that

$$\|y\|_2^2 \leq \|\theta\|_2^2. \quad (81)$$

Thus, the energy of the system's output is bounded from above by the energy of the constraint process.

The minimum-variance filter and smoother which produce estimates of a linear system's output, minimise the mean square error. Here, it is desired to calculate estimates that trade off minimum mean-square-error performance and achieve

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“Man's greatest asset is the unsettled mind.” *Isaac Asimov*

$$\|\hat{y}\|_2^2 \leq \|\theta\|_2^2. \quad (82)$$

Note that (80) implies (81) but the converse is not true. Although estimates  $\hat{y}_{j,k}$  of  $y_{j,k}$  satisfying  $-\theta_{j,k} \leq \hat{y}_{j,k} \leq \theta_{j,k}$  are desirable, the procedures described below only ensure that (82) is satisfied.

### 10.3.3 Constrained Filtering

A procedure is proposed in which a linear filter  $\mathcal{H}: \mathbb{R}^p \rightarrow \mathbb{R}^p$  is used to calculate estimates  $\hat{y}$  from zero-mean measurements  $z_k$  that are constrained using an odd censoring function to obtain

$$\underline{z}_k = \begin{bmatrix} z_{1,k} \\ \vdots \\ z_{p,k} \end{bmatrix} = \begin{bmatrix} g_o(z_{1,k}, \gamma^{-1}\theta_{1,k}) \\ \vdots \\ g_o(z_{p,k}, \gamma^{-1}\theta_{p,k}) \end{bmatrix}, \quad (83)$$

which satisfy

$$\|\underline{z}\|_2^2 \leq \gamma^{-2} \|\theta\|_2^2. \quad (84)$$

where  $\underline{z} = [\underline{z}_1^T \dots \underline{z}_N^T]^T$ , for a positive  $\gamma \in \mathbb{R}$  to be designed. This design problem is depicted in Fig. 10.

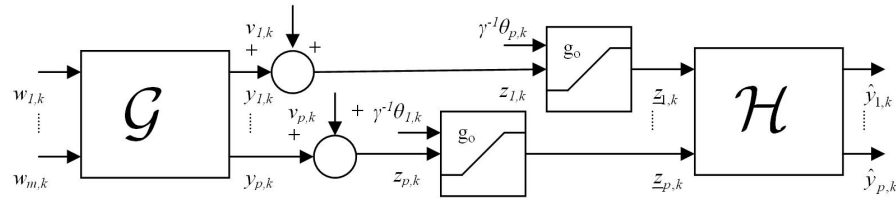


Fig. 10. The constrained filtering design problem. The task is to design a scalar  $\gamma$  so that the outputs of a filter  $\mathcal{H}$  operating on the censored zero-mean measurements  $[\underline{z}_{1,k}^T \dots \underline{z}_{p,k}^T]^T$  produce output estimates  $[\hat{y}_{1,k}^T \dots \hat{y}_{p,k}^T]^T$ , which trade off mean square error performance and achieve  $\|\hat{y}\|_2^2 \leq \|\theta\|_2^2$ .

Censoring the measurements is suggested as a low-implementation-cost approach to constrained filtering. Design constraints are sought for the measurement

censoring functions so that the outputs of a subsequent filter satisfy the performance objective (82). The recursions akin to the minimum-variance filter are applied to calculate predicted and filtered state estimates from the constrained measurements  $\underline{z}_k$  at time  $k$ . That is, the output mapping  $C_k$  is retained within the linear filter design even though nonlinearities are present with (83). The predicted states, filtered states and output estimates are respectively obtained as

$$\hat{x}_{k+1/k} = (A_k - K_k C_k) \hat{x}_{k/k-1} + K_k \underline{z}_k, \quad (85)$$

$$\hat{x}_{k/k} = (I - L_k C_k) \hat{x}_{k/k-1} + L_k \underline{z}_k, \quad (86)$$

$$\hat{y}_{k/k} = C_k \hat{x}_{k/k}, \quad (87)$$

where  $L_k = P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1}$ ,  $K_k = A_k L_k$ , and  $P_{k/k-1} = P_{k/k-1}^T > 0$  is obtained from  $P_{k/k} = P_{k/k-1} - P_{k/k-1} C_k^T (C_k P_{k/k-1} C_k^T + R_k)^{-1} C_k P_{k/k-1}$ ,  $P_{k+1/k} = A_k P_{k/k} A_k^T + B_k Q_k B_k^T$ . Nonzero-mean sequences can be accommodated using deterministic inputs as described in Chapter 4. Since a nonlinear system output (79) and a nonlinear measurement (83) are assumed, the estimates calculated from (85) – (87) are not optimal. Some properties that are exhibited by these estimates are described below.

**Lemma 3 [19]:** *In respect of the filter (85) – (87) which operates on the constrained measurements (83), suppose the following:*

- (i) *the probability density functions associated with  $w_k$  and  $v_k$  are even;*
- (ii) *the nonlinear functions within (79) and (83) are odd; and*
- (iii) *the filter is initialised with  $\hat{x}_{0/0} = E\{x_0\}$ .*

*Then the following applies:*

- (i) *the predicted state estimates,  $\hat{x}_{k+1/k}$ , are unbiased;*
- (ii) *the corrected state estimates,  $\hat{x}_{k/k}$ , are unbiased; and*
- (iii) *the output estimates,  $\hat{y}_{k/k}$ , are unbiased.*

**Proof:** (i) Condition (iii) implies  $E\{\tilde{x}_{1/0}\} = 0$ , which is the initialisation step of an induction argument. It follows from (85) that

$$\hat{x}_{k+1/k} = (A_k - K_k C_k) \hat{x}_{k/k-1} + K_k (C_k x_k + v_k) + K_k (\underline{z}_k - C_k x_k - v_k). \quad (88)$$

Subtracting (88) from (78) gives  $\tilde{x}_{k+1/k} = (A_k - K_k C_k) \tilde{x}_{k/k-1} - B_k w_k - K_k v_k - K_k (\underline{z}_k - C_k x_k - v_k)$  and therefore

$$E\{\tilde{x}_{k+1/k}\} = (A_k - K_k C_k) E\{\tilde{x}_{k/k-1}\} + B_k E\{w_k\} - K_k E\{v_k\} - K_k E\{\underline{z}_k - C_k x_k - v_k\}. \quad (89)$$

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“All truth passes through three stages: First, it is ridiculed; Second, it is violently opposed; and Third, it is accepted as self-evident.” *Arthur Schopenhauer*

From above assumptions, the second and third terms on the right-hand-side of (89) are zero. The property (77) implies  $E\{\underline{z}_k\} = E\{z_k\} = E\{C_k x_k + v_k\}$  and so  $E\{\underline{z}_k - C_k x_k - v_k\}$  is zero. The first term on the right-hand-side of (89) pertains to the unconstrained Kalman filter and is zero by induction. Thus,  $E\{\tilde{x}_{k+1/k}\} = 0$ .

(ii) Condition (iii) again serves as an induction assumption. It follows from (86) that

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k(C_k x_k + v_k - C_k \hat{x}_{k/k-1}) + L_k(\underline{z}_k - C_k x_k - v_k). \quad (90)$$

Substituting  $x_k = A_{k-1}x_{k-1} + B_{k-1}w_{k-1}$  into (90) yields  $\tilde{x}_{k/k} = (I - L_k C_k)A_{k-1}\tilde{x}_{k-1/k-1} + (I - L_k C_k)B_{k-1}w_{k-1} - L_k v_k - L_k(\underline{z}_k - C_k x_k - v_k)$  and  $E\{\tilde{x}_{k/k}\} = (I - L_k C_k)A_{k-1}E\{\tilde{x}_{k-1/k-1}\} = (I - L_k C_k)A_{k-1} \dots (I - L_1 C_1)A_0 E\{\tilde{x}_{0/0}\}$ . Hence,  $E\{\tilde{x}_{k/k}\} = 0$  by induction.

$$\begin{aligned} \text{(iv)} \quad \text{Defining } \tilde{y}_{k/k} &= y_k - \hat{y}_{k/k} = y_k + C_k(x_k - \hat{x}_{k/k}) - C_k x_k = \\ &= C_k \tilde{x}_{k/k} + y_k - C_k x_k \text{ and using (77) leads to } E\{\tilde{y}_{k/k}\} = C_k E\{\tilde{x}_{k/k}\} \\ &+ E\{y_k - C_k x_k\} = C_k E\{\tilde{x}_{k/k}\} = 0 \text{ under condition (iii).} \end{aligned}$$

(v)

Recall that the Bounded Real Lemma (see Lemma 7 of Chapter 9) specifies a bound for a ratio of a system's output and input energies. This lemma is used to find a design for  $\gamma$  within (83) as described below.

**Lemma 4 [19]:** Consider the filter (85) – (87) which operates on the constrained measurements (83). Let  $\underline{A}_k = A_k - K_k C_k$ ,  $\underline{B}_k = K_k$ ,  $\underline{C}_k = C_k(I - L_k C_k)$  and  $\underline{D}_k = C_k L_k$  denote the state-space parameters of the filter. Suppose for a given  $\gamma_2 > 0$ , that a solution  $M_k = M_k^T > 0$  exists over  $k \in [1, N]$  for the Riccati Difference equation resulting from the application of the Bounded Real Lemma to the system  $\begin{bmatrix} \underline{A}_k & \underline{B}_k \\ \underline{C}_k & \underline{D}_k \end{bmatrix}$ . Then the design  $\gamma = \gamma_2$  within (83) results in the performance objective (82) being satisfied.

**Proof:** For the application of the Bounded Real Lemma to the filter (85) – (87), the existence of a solution  $M_k = M_k^T > 0$  for the associated Riccati difference equation ensures that  $\|\hat{y}\|_2^2 \leq \gamma_2^2 \|\underline{z}\|_2^2 - x_0^T M_0 x_0 \leq \gamma_2^2 \|\underline{z}\|_2^2$ , which together with (84) leads to (82).

It is argued below that the proposed filtering procedure is asymptotically stable.

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“Everything we know is only some kind of approximation, because we know that we do not know all the laws yet. Therefore, things must be learned only to be unlearned again or, more likely, to be corrected.” *Richard Phillips Feynman*



**Lemma 5 [19]:** Define the filter output estimation error as  $\tilde{y} = y - \hat{y}$ . Under the conditions of Lemma 4,  $\tilde{y} \in \ell_2$ .

**Proof:** It follows from  $\tilde{y} = y - \hat{y}$  that  $\|\tilde{y}\|_2 \leq \|y\|_2 + \|\hat{y}\|_2$ , which together with (10) and the result of Lemma 4 yields  $\|\tilde{y}\|_2 \leq 2\|\theta\|_2$ , thus the claim follows.

### 10.3.4 Constrained Smoothing

In the sequel, it is proposed that the minimum-variance fixed-interval smoother recursions operate on the censored measurements  $\underline{z}_k$  to produce output estimates  $\hat{y}_{k/N}$  of  $y_k$ .

**Lemma 6 [19]:** In respect of the minimum-variance smoother recursions that operate on the censored measurements  $\underline{z}_k$ , under the conditions of Lemma 3, the smoothed estimates,  $\hat{y}_{k/N}$ , are unbiased.

The proof follows *mutatis mutandis* from the approach within the proofs of Lemma 5 of Chapter 7 and Lemma 3. An analogous result to Lemma 5 is now stated.

**Lemma 7 [19]:** Define the smoother output estimation error as  $\tilde{y} = y - \hat{y}$ . Under the conditions of Lemma 3,  $\tilde{y} \in \ell_2$ .

The proof follows *mutatis mutandis* from that of Lemma 5. Two illustrative examples are set out below. A GPS and inertial navigation system integration application is detailed in [19].

**Example 5 [19].** Consider the saturating nonlinearity

$$g_o(X, \beta) = 2\beta\pi^{-1} \arctan(\pi X(2\beta)^{-1}). \quad (91)$$

which is a continuous approximation of (76) that satisfies  $|g_o(X, \beta)| \leq |\beta|$  and  $\frac{dg_o(X, \beta)}{dX} = (1 + (\pi X)^2(2\beta)^{-2})^{-1} \approx 1$  when  $(\pi X)^2(2\beta)^{-2} \ll 1$ . Data was

generated from (78), (79), (91), where  $A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}$ ,  $B = C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,

Gaussian, white, zero-mean processes with  $Q = R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$ . The constraint

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“A man whose errors take ten years to correct is quite a man.” *Julius Robert Oppenheimer*

vector within (80) was chosen to be fixed, namely,  $\theta_k = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ ,  $k \in [1, 10^5]$ . The

limits of the observed distribution of estimates,  $\hat{y}_{k/k} = \begin{bmatrix} \hat{y}_{1,k/k} \\ \hat{y}_{2,k/k} \end{bmatrix}$ , arising by

operating the minimum-variance filter recursions on the raw data  $z_k = y_k + v_k$  are indicated by the outer black region of Fig. 11. It can be seen that the filter outputs do not satisfy the performance objective (82), which motivates the pursuit of constrained techniques. A minimum value of  $\gamma_2 = 1.24$  was found for the solutions of the Riccati difference equation mentioned specified within Lemma 4 to be positive definite. The filter (85) – (87) was applied to the censored measurements

$$\underline{z}_k = \begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = \begin{bmatrix} g_o(z_{1,k}, \gamma^{-1} \theta_{1,k}) \\ g_o(z_{2,k}, \gamma^{-1} \theta_{2,k}) \end{bmatrix} \text{ using (91). The limits of the observed}$$

distribution of the constrained filter estimates are indicated by the inner white region of Fig. 11. The figure shows that the constrained filter estimates satisfy (82), which illustrates Lemma 5.

**Example 6 [19].** Measurements were similarly synthesised using the parameters of Example 5 to demonstrate constrained fixed-interval smoother performance. A minimum value of  $\gamma_2 = 5.6$  was found for the solutions of the Riccati difference equation mentioned within Lemma 4 to be positive definite. The superimposed distributions of the unconstrained and constrained smoothers are respectively indicated by the inner and outer black regions of Fig. 12. It can be seen by inspection of the figure that the constrained smoother estimates meets (80), whereas those produced by the standard smoother do not.

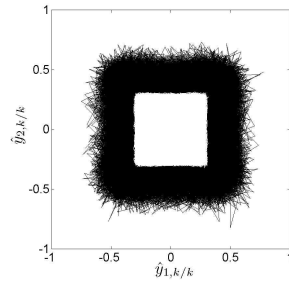


Fig. 11. Superimposed distributions of filtered estimates for Example 4: unconstrained filter (outer black); and constrained filter (middle white).

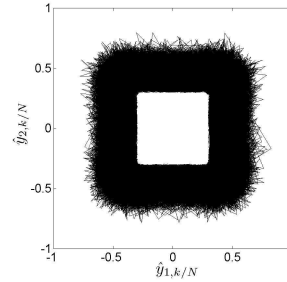


Fig. 12. Superimposed distributions of smoothed estimates for Example 5: unconstrained smoother (outer black); and constrained smoother (middle white).

The above examples involved searching for minimum value of  $\gamma_2$  for the existence of positive definite solutions for the Riccati equation alluded to within Lemma 4.

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“An expert is a man who has made all the mistakes which can be made in a very narrow field.” *Niels Henrik David Bohr*

The need for a search may not be apparent as stability is guaranteed whenever a positive definite solution for the associated Riccati equation exists. Searching for a minimum  $\gamma_2$  is advocated because the use of an excessively large value can lead to a nonlinearity design that is conservative and exhibits poor mean-square-error performance. If a design is still too conservative then an empirical value, namely,  $\gamma_2 = \|\hat{y}\|_2 \|\underline{z}\|_2^{-1}$ , may need to be considered instead.

## 10.6 Chapter Summary

In this chapter it is assumed that nonlinear systems are of the form  $x_{k+1} = a_k(x_k) + b_k(w_k)$ ,  $y_k = c_k(x_k)$ , where  $a_k(\cdot)$ ,  $b_k(\cdot)$  and  $c_k(\cdot)$  are continuous differentiable functions. The EKF arises by linearising the model about conditional mean estimates and applying the standard filter recursions. The first, second and third-order EKFs simplified for the case of  $x_k \in \mathbb{R}$  are summarised in Table 1.

The EKF attempts to produce locally optimal estimates. However, it is not necessarily stable because the solutions of the underlying Riccati equations are not guaranteed to be positive definite. The faux algebraic Riccati technique trades off approximate optimality for stability. The familiar structure of the EKF is retained but stability is achieved by selecting a positive definite solution to a faux Riccati equation for the gain design.

$H_\infty$  techniques can be used to recast nonlinear filtering applications into a model uncertainty problem. It is demonstrated with the aid of an example that a robust EKF can reduce the mean square error when the problem is sufficiently nonlinear.

Linearised models may be applied within the previously-described smoothers in the pursuit of performance improvement. Nonlinear versions of the fixed-lag, Fraser-Potter and Rauch-Tung-Striebel smoothers are easier to implement as they are less complex. However, the application of the minimum-variance smoother can yield approximately optimal estimates when the problem becomes linear, provided that the underlying assumptions are correct. A smoother that is robust to input uncertainty is obtained by replacing the approximate error covariance correction with an  $H_\infty$  version. The resulting robust nonlinear smoother can exhibit performance benefits when uncertainty is present.

In some applications, it may be possible to censor a system's inputs, states or outputs, rather than proceed with an EKF design. It has been shown that the use of a nonlinear censoring function to constrain input measurements leads to bounded filter and smoother estimation errors.

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“Most of what I learned as an entrepreneur was by trial and error.” *Gordon Earl Moore*

	Linearisation	Predictor-Corrector Form
1 <sup>st</sup> -order EKF	$A_k = \left. \frac{\partial a_k(x)}{\partial x} \right _{x=\hat{x}_{k/k}}$ $C_k = \left. \frac{\partial c_k(x)}{\partial x} \right _{x=\hat{x}_{k/k-1}}$ $B_k = b_k(\hat{x}_{k/k})$	$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k(z_k - c_k(\hat{x}_{k/k-1}))$ $\hat{x}_{k+1/k} = a_k(\hat{x}_{k/k})$
2 <sup>nd</sup> -order EKF	$A_k = \left. \frac{\partial a_k(x)}{\partial x} \right _{x=\hat{x}_{k/k}}$ $C_k = \left. \frac{\partial c_k(x)}{\partial x} \right _{x=\hat{x}_{k/k-1}}$ $B_k = b_k(\hat{x}_{k/k})$	$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L_k \left( z_k - c_k(\hat{x}_{k/k-1}) - \frac{1}{2} \frac{\partial^2 c_k}{\partial x^2} \bigg _{x=\hat{x}_{k/k-1}} \right)$ $\hat{x}_{k+1/k} = a_k(\hat{x}_{k/k}) + \frac{1}{2} P_{k/k} \frac{\partial^2 a_k}{\partial x^2} \bigg _{x=\hat{x}_{k/k}}$
3 <sup>rd</sup> -order EKF	$A_k = \left. \frac{\partial a_k(x)}{\partial x} \right _{x=\hat{x}_{k/k}} + \frac{1}{6} P_{k/k} \frac{\partial^2 a_k}{\partial x^2} \bigg _{x=\hat{x}_{k/k}}$ $C_k =$ $B_k =$	

Table 1. Summary of first, second and third-order EKFs for the case of  $x_k \in \cdot$ .

## 10.7 Problems

**Problem 1.** Use the following Taylor series expansion of  $f(x)$

$$\begin{aligned}
 f(x) = f(x_0) &+ \frac{1}{1!}(x-x_0)^T \nabla f(x_0) + \frac{1}{2!}(x-x_0)^T \nabla^T \nabla f(x_0)(x-x_0) \\
 &+ \frac{1}{3!}(x-x_0)^T \nabla^T \nabla (x-x_0) \nabla f(x_0)(x-x_0) \\
 &+ \frac{1}{4!}(x-x_0)^T \nabla^T \nabla (x-x_0) \nabla (x-x_0) \nabla f(x_0)(x-x_0) + \dots,
 \end{aligned}$$

to find expressions for the coefficients  $\alpha_i$  within the functions below.

- (i)  $f(x) = \alpha_0 + \alpha_1(x-x_0) + \alpha_2(x-x_0)^2$ .
- (ii)  $f(x) = \alpha_0 + \alpha_1(x-x_0) + \alpha_2(x-x_0)^2 + \alpha_3(x-x_0)^3$ .
- (iii)  $f(x, y) = \alpha_0 + \alpha_1(x-x_0) + \alpha_2(x-x_0)^2$   
 $+ \alpha_3(y-y_0) + \alpha_4(y-y_0)^2 + \alpha_5(x-x_0)(y-y_0)$
- (iv)  $f(x, y) = \alpha_0 + \alpha_1(x-x_0) + \alpha_2(x-x_0)^2 + \alpha_3(x-x_0)^3$

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“The capacity to blunder slightly is the real marvel of DNA. Without this special attribute, we would still be anaerobic bacteria and there would be no music.” *Lewis Thomas*

$$\begin{aligned}
& +\alpha_4(y-y_0)+\alpha_5(y-y_0)^2+\alpha_6(y-y_0)^3 \\
& +\alpha_7(x-x_0)(y-y_0)+\alpha_8(x-x_0)^2(y-y_0) \\
& +\alpha_9(x-x_0)(y-y_0)^2. \\
\text{(v)} \quad f(x,y) & =\alpha_0+\alpha_1(x-x_0)+\alpha_2(x-x_0)^2+\alpha_3(x-x_0)^3+\alpha_4(x-x_0)^4 \\
& +\alpha_5(y-y_0)+\alpha_6(y-y_0)^2+\alpha_7(y-y_0)^3+\alpha_8(y-y_0)^4 \\
& +\alpha_9(x-x_0)(y-y_0)+\alpha_{10}(x-x_0)^2(y-y_0) \\
& +\alpha_{11}(x-x_0)(y-y_0)^2+\alpha_{12}(x-x_0)^3(y-y_0) \\
& +\alpha_{13}(x-x_0)(y-y_0)^3+\alpha_{14}(x-x_0)^2(y-y_0)^2.
\end{aligned}$$

**Problem 2.** Consider a state estimation problem, where  $x_{k+1} = a_k(x_k) + B_k w_k$ ,  $y_k = c_k(x_k)$ ,  $z_k = y_k + v_k$ , in which  $w_k, x_k, y_k, v_k, a_k(\cdot), B_k, c_k(\cdot) \in \mathbb{R}$ . Derive the

- (i) first-order,
- (ii) second-order,
- (iii) third-order and
- (iv) fourth-order EKF's,

assuming the required derivatives exist.

**Problem 3.** Suppose that an FM signal is generated by  $a_{k+1} = \mu_a a_k + w_k^{(1)}$ ,  $\omega_{k+1} = \mu_\omega \omega_k + w_k^{(2)}$ ,  $\phi_{k+1} = \phi_k + \omega_k$ ,  $z_k^{(1)} = a_k \cos(\phi_k) + v_k^{(1)}$  and  $z_k^{(2)} = a_k \sin(\phi_k) + v_k^{(2)}$ .

Write down the recursions for

- (i) first-order and
- (ii) second-order

EKF demodulators.

**Problem 4. (Continuous-time EKF)** Assume that continuous-time signals may be modelled as  $\dot{x}(t) = a(x(t)) + w(t)$ ,  $y(t) = c(x(t))$ ,  $z(t) = y(t) + v(t)$ , where  $E\{w(t)w^T(t)\} = Q(t)$  and  $E\{v(t)v^T(t)\} = R(t)$ .

- (i) Show that approximate state estimates can be obtained from  $\dot{\hat{x}}(t) = a(\hat{x}(t)) + K(t)(z(t) - c(\hat{x}(t)))$ , where  $K(t) = P(t)C^T(t)R^{-1}(t)$ ,  $\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - K(t)C(t)P(t) + Q(t)$ ,  $A(t) = \left. \frac{\partial a(x)}{\partial x} \right|_{x=x(t)}$  and  $C(t) = \left. \frac{\partial c(x)}{\partial x} \right|_{x=x(t)}$ .
- (ii) Often signal models are described in the above continuous-time setting but sampled measurements  $z_k$  of  $z(t)$  are available. Write down a hybrid continuous-discrete version of the EKF in corrector-predictor form.

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“I am quite conscious that my speculations run quite beyond the bounds of true science.” *Charles Robert Darwin*

**Problem 5.** Consider a pendulum of length  $\ell$  that subtends an angle  $\theta(t)$  with a vertical line through its pivot. The pendulum's angular acceleration and measurements of its instantaneous horizontal position (from the vertical) may be

modelled as  $\frac{d^2\theta(t)}{dt^2} = -\frac{g}{\ell}\sin(\theta(t)) + w(t)$  and  $z(t) = \ell\sin(\theta(t)) + v(t)$ ,

respectively, where  $g$  is the gravitational constant,  $w(t)$  and  $v(t)$  are stochastic inputs.

- (i) Set out the pendulum's equations of motion in a state-space form and write down the continuous-time EKF for estimating  $\theta(t)$  from  $v(t)$ .
- (ii) Use Euler's first-order integration formula to discretise the above model and then detail the corresponding discrete-time EKF.

## 10.8 Glossary

$\nabla f$	The gradient of a function $f$ , which is a row-vector of partial derivatives.
$\nabla^T \nabla f$	The Hessian of a function $f$ , which is a matrix of partial derivatives.
$\text{tr}(P_k)$	The trace of a matrix $P_k$ , which is the sum of its diagonal terms.
FM	Frequency modulation.

## 10.9 References

- [1] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York, 1971.
- [2] A. Gelb, *Applied Optimal Estimation*, The Analytic Sciences Corporation, USA, 1974.
- [3] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1979.
- [4] T. Söderström, *Discrete-time Stochastic Systems: Estimation and Control*, Springer-Verlag London Ltd., 2002.
- [5] D. Simon, *Optimal State Estimation, Kalman  $H_\infty$  and Nonlinear Approaches*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.
- [6] R. R. Bitmead, A.-C. Tsoi and P. J. Parker, "Kalman filtering approach to short time Fourier analysis", *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 34, no. 6, pp. 1493 – 1501, Jun. 1986.
- [7] M.-A. Poubelle, R. R. Bitmead and M. Gevers, "Fake Algebraic Riccati Techniques and Stability", *IEEE Transactions on Automatic Control*, vol. 33, no. 4, pp. 379 – 381, Apr. 1988.

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"What we observe is not nature itself, but nature exposed to our mode of questioning." *Werner Heisenberg*

- [8] R. R. Bitmead, M. Gevers and V. Wertz, *Adaptive Optimal Control. The thinking Man's GPC*, Prentice Hall, New York, 1990.
- [9] R. R. Bitmead and Michel Gevers, "Riccati Difference and Differential Equations: Convergence, Monotonicity and Stability", In S. Bittanti, A. J. Laub and J. C. Willems (Eds.), *The Riccati Equation*, Springer Verlag, 1991.
- [10] G. A. Einicke, L. B. White and R. R. Bitmead, "The Use of Fake Algebraic Riccati Equations for Co-channel Demodulation", *IEEE Transactions on Signal Processing*, vol. 51, no. 9, pp. 2288 – 2293, Sep., 2003.
- [11] C. A. Desoer and M. Vidyasagar, *Feedback Systems : Input Output Properties*, Academic Press, NewYork, 1975.
- [12] G. A. Einicke and L. B. White, "Robust Extended Kalman Filtering", *IEEE Transactions on Signal Processing*, vol. 47, no. 9, pp. 2596 – 2599, Sep., 1999.
- [13] J. Aisbett, "Automatic Modulation Recognition Using Time Domain Parameters", *Signal Processing*, vol. 13, pp. 311-323, 1987.
- [14] P. S. Maybeck, *Stochastic models, estimation, and control*, Academic Press, New York, vol. 1, 1979.
- [15] H. E. Doran, "Constraining Kalman filter and smoothing estimates to satisfy time-varying restrictions", *Review of Economics and Statistics*, vol. 74, no. 3, pp. 568 – 572, 1992.
- [16] D. Massicotte, R. Z. Morawski and A. Barwicz, "Incorporation of a Positivity Constraint Into A Kalman-Filter-Based Algorithm for Correction of Spectrometric Data", *IEEE Transactions on Instrumentation and Measurement*, vol. 44, no. 1, pp. 2 – 7, 1995.
- [17] D. Simon and T. L. Chia, "Kalman Filtering with State Equality Constraints", *IEEE Transactions on Aerospace and Electronic Systems*, vol. 38, no. 1, pp. 128 – 136, 2002.
- [18] S. J. Julier and J. J. LaViola, "On Kalman Filtering Within Nonlinear Equality Constraints", *IEEE Transactions on Signal Processing*, vol. 55, no. 6, pp. 2774 – 2784, Jun. 2007.
- [19] G. A. Einicke, G. Falco and J. T. Malos, "Bounded Constrained Filtering for GPS/INS Integration", *IEEE Transactions on Automated Control*, 2012 (to appear).

## 11. Hidden Markov Model Filtering and Smoothing

### 11.1 Introduction

The previously-discussed optimal Kalman filter [1] – [3] is routinely used for tracking observed and unobserved states whose second-order statistics change over time. It is often assumed within Kalman filtering applications that one or more random variable sequences are generated by a random walk or an autoregressive process. That is, common Kalman filter parameterisations do not readily exploit knowledge about the random variables’ probability distributions. More precisely, the filter is optimal only for Gaussian variables whose first and second order moments completely specify all relevant probability distributions. For non-Gaussian data, the filter is only optimal over all *linear* filters [1].

Rather than assuming that random variable sequences are generated by autoregressive processes they may alternatively be modelled as Markov chains. The phrase ‘Markov chain’ was first coined in 1926 by a Russian mathematician S. N. Bernstein to acknowledge previous discoveries made by Andrei Andreevich Markov [4]. Markov was a professor at St Petersburg University and a member of the St Petersburg Academy of Sciences, which was a hub for scientific advances in many fields including probability theory. Indeed, Markov, along with fellow academy members D. Bernoulli, V. Y. Bunyakovsky and P. L. Chebyshev, all wrote textbooks on probability theory. Markov extended the weak law of large numbers and the central limit theorem to certain sequences of dependent random variables forming special classes of what are now known as Markov chains [4].

The basic theory of Hidden Markov models (HMMs) was first published by Baum *et al* in the 1960s [5]. HMMs were introduced to the speech recognition field in the 1970s by J. Baker at CMU [6], and F. Jelinek and his colleagues at IBM [7]. One of the most influential papers on HMM filtering and smoothing was the tutorial exposition by L. Rabiner [8], which has been accorded a large number of citations. Rabiner explained how to implement the forward-backward algorithm for estimating Markov state probabilities, together with the Baum-Welch algorithm (also known as the Expectation Maximisation algorithm). HMM filters

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“The alleged opinion that studies in seminars are of the highest scientific nature, while exercises in solving problems are of the lowest rank, is unfair. Mathematics to a considerable extent consists of solving problems, and together with proper discussion, this can be of the highest scientific nature.”  
*Andrey Andreyevich Markov*



and smoothers can be advantageous in applications where sequences of alphabets occur [8] - [10]. For example, in automatic speech recognition, sentence and language models can be constructed by concatenating phoneme and word-level HMMs. Similarly, stroke, character, word and context HMMs can be used in handwriting recognition. HMMs have been useful in modelling in biological sequences such as proteins and DNA sequences.

The Doob–Meyer decomposition theorem [11] states that a stochastic process may be decomposed into the sum of two parts, namely, a prediction and an input process. The standard Kalman filter [1] makes use of *both* prediction plus input process assumptions and attains minimum-variance optimality. In contrast, the standard hidden Markov model filter/smoothers rely exclusively on (Markov model) prediction and is optimum in a Bayesian sense [8] - [10]. It is shown below that minimum-variance and HMM techniques can be combined for improved state recovery.

The minimum-variance, HMM and combined-minimum-variance-HMM predictions are only calculated from states at the previous time step. Improved predictions can be calculated from states at *multiple* previous time steps. The desired interdependencies between multiple previous states are conveniently captured by constructing high-order-Kronecker-product state vectors. The theory and implementation of such high-order-minimum-variance-HMM filters is also described below.

The afore-mentioned developments are driven by our rapacious appetites for improved estimator performance. In principle, each additional embellishment, spanning HMM filters, minimum-variance-HMM filters to high-order-minimum-variance-HMM filters, has potential to provide further performance gains, subject to the usual proviso that the underlying modelling assumptions are correct. Needless to say, significantly higher calculation overheads must be reconciled against any performance benefits.

Some prerequisites, namely, some results from probability theory including Markov processes, are introduced in Section 11.2. Bayes' theorem is judiciously applied in Section 11.3 to derive the HMM filters and smoothers for time-homogenous processes. A state-space model having an output covariance equivalent to an HMM is derived in Section 11.4. This enables transition probability matrices to be employed in optimal filter and smoother constructions that minimise the error variance. Section 11.5 describes high-order-minimum-variance-HMM filters, which employ Kronecker product states.

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“It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge.” *Pierre Simon Laplace*

## 11.2 Prerequisites

### 11.2.1 Bayes' Theorem

Bayes' theorem (also known as Bayes' rule) is used to update the probability of an event  $A$  in the light of new evidence  $B$ . Let  $\Pr\{A\}$  and  $\Pr\{B\}$  denote the (prior) probabilities of  $A$  and  $B$  occurring independently of each other. The (posterior) conditional probability of  $A$  given that  $B$  has happened can be expressed as

$$\Pr\{A | B\} = \frac{\Pr\{A\} \Pr\{B | A\}}{\Pr\{B\}},$$

where  $\Pr\{B|A\}$  is the conditional probability of  $B$  occurring given that  $A$  has occurred.

### 11.2.2 Joint Probabilities

The probability of event  $A$  and event  $B$  occurring is given by

$$\begin{aligned} \Pr\{A, B\} &= \Pr\{A | B\} \Pr\{B\} \\ &= \Pr\{B | A\} \Pr\{A\}. \end{aligned}$$

The probability of events  $A$ ,  $B$  and  $C$  occurring is given by

$$\begin{aligned} \Pr\{A, B, C\} &= \Pr\{A | B, C\} \Pr\{B, C\} \\ &= \Pr\{A | B, C\} \Pr\{B | C\} \Pr\{C\}. \end{aligned}$$

### 11.2.3 Total Probability Theorem

The Total Probability Theorem (also known as the Law of Total Probability) breaks up probability calculations into parts. Given  $n$  mutually exclusive events  $B_1, \dots, B_n$  whose probabilities  $\Pr\{B_1\}, \dots, \Pr\{B_n\}$  sum to zero, then the probability of  $A$  occurring is given by

$$\Pr\{A\} = \Pr\{A | B_1\} \Pr\{B_1\} + \dots + \Pr\{A | B_n\} \Pr\{B_n\},$$

where  $\Pr\{A|B_i\}$  is the conditional probability of  $A$  occurring given that  $B_i$  has occurred.

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“The combination of Bayes' and Markov Chain Monte Carlo has been called arguably the most powerful mechanism ever created for processing data and knowledge.” *Sharon Bertsch McGrayne, The Theory That would Not Die: How Bayes' Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two centuries of Controversy*

### 11.2.4 Markov Chain

The terms *stochastic process* and *random process* are often used interchangeably and so it is appropriate to commence with some formal definitions.

**Definition 1.** A discrete stochastic process is a sequence of events in which the outcome at any stage depends on some probability.

**Definition 2.** A discrete Markov process is a stochastic process with the following properties:

- (a) The number of possible outcomes is finite;
- (b) The outcome at the next stage depends only on the outcome of the current stage and not on the previous stages.

**Assumption 1 [8]:** Let  $x_k$  be a finite state, discrete-time, first-order Markov process, that takes on (or has been quantised into) an alphabet of  $n$  discrete values in a set, *i.e.*,

$$x_k \in \{q_1, \dots, q_n\}, q_i \in \mathbb{R}. \quad (1)$$

The probability that  $x_k$  takes on value  $q_i$  is denoted by  $\pi_k(i) = \Pr\{x_k = q_i\} \geq 0$ . It is convenient to stack these probabilities into a so-called probability distribution vector as  $\pi_k = [\pi_k(1), \pi_k(2), \dots, \pi_k(n)]^T \in \mathbb{R}^n$  in which  $\sum_{i=1}^n \pi_k(i) = 1$ . The attention is confined to stationary (or time-homogenous) stochastic processes, in which the probabilities are constant over time. In particular, it is assumed that the probability distribution vector states evolve according to

$$\pi_{k+1} = A\pi_k, \quad (2)$$

for all  $k \in [1, \dots, N]$ , where  $A = \{a_{i,j}\} \in \mathbb{R}^{n \times n}$ ,  $a_{i,j} = \Pr\{x_{k+1} = q_i | x_k = q_j\}$ , is a column-stochastic transition probability matrix for  $\pi_k$  satisfying  $a_{i,j} \geq 0$ ,  $\sum_{j=1}^n a_{i,j} = 1$ . A sequence or chain of states  $\pi_k$  generated by (2) is known as a first-order, discrete Markov chain. Suppose also that  $x_k$  is hidden and observed indirectly by measurements

$$y_k = x_k + v_k, \quad (3)$$

where  $v_k$  is a white, Gaussian measurement noise process. Equations (1) – (3) are commonly referred to as an HMM.

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“In science, progress is possible. In fact, if one believes in Bayes’ theorem, scientific progress is inevitable as predictions are made and as beliefs are tested and refined.” *Nat Silver*

Note that the column and row indices of  $A$  pertain to the next and current probability distribution states, respectively, *i.e.*,

$$A = \begin{matrix} & \text{next state} \\ \text{current state} & \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{N,1} & \cdots & a_{n,n} \end{bmatrix} \end{matrix}.$$

Let  $\pi_1$  denote an initial probability distribution vector of the Markov chain. Then repeated application of (2) yields  $\pi_{k+1} = A^{k-1}\pi_1$ ,  $k > 1$ . If the Markov chain is non-absorbing or irreducible (it can go from any state to any state) and is non-periodic, it has a stationary (or limiting) distribution denoted by  $\pi = \lim_{k \rightarrow \infty} \pi_k$ . The stationary distribution, which satisfies  $\pi = A\pi$ , is a useful quantity, and can be found by singular value decomposition.

*Example 1:* Suppose that a Markov process  $x_k$ ,  $k = 1, \dots, 10$ , has the sequence 1, 1, 2, 2, 2, 1, 1, 2, 2, 2. By inspection  $\Pr\{x_{k+1}=1|x_k=1\} = 0.5$ ,  $\Pr\{x_{k+1}=2|x_k=1\} = 0.5$ ,  $\Pr\{x_{k+1}=1|x_k=2\} = 0.2$  and  $\Pr\{x_{k+1}=2|x_k=2\} = 0.8$ , which yields  $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$ .

The columns of  $A$  sum to unity (by construction). It is easily verified that the stationary distribution is  $\pi = [0.29, 0.71]^T$ .

## 11.3 HMM Filter and Smoother Derivation

### 11.3.1 HMM Filter

It is convenient to temporarily make the following simplifying assumption.

**Assumption 2 [9] – [8]:** Without loss of generality, temporarily assume that  $x_k$  is a finite state, discrete-time, first-order Markov process, that takes on integer index values

$$x_k \in \{1, \dots, n\}. \quad (4)$$

The corresponding transition probability matrix in (2) is now  $A = \{a_{i,j}\} \in \mathbb{R}^{n \times n}$ ,  $a_{i,j} = \Pr\{x_{k+1}=i | x_k=j\}$ . Assume again that indirect measurements (3) are available.

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"I always avoid prophesying beforehand because it is much better to prophesy after the event has already taken place." *Sir Winston Leonard Spencer-Churchill*

The filtering objective is to find optimal estimates  $\hat{x}_k \in \{1, \dots, n\}$  of the hidden states  $x_k$ , for the measurement sequence  $y_1$  to  $y_k$ . From the approach of [8], suitable estimates are found from the *a posteriori* state probabilities at time  $k$  and observations  $y_1$  to  $y_k$ ,

$$\alpha_k(i) = \Pr\{x_k = i, y_1, \dots, y_k\}. \quad (5)$$

The cost of evaluating (5) grows exponentially with  $k$  and therefore a recursive expression for  $\alpha_k$  is derived below. Applying the law of total probability and the joint probabilities formula to (5) then simplifying using the Markov property for observations yields

$$\begin{aligned} \alpha_k(i) &= \sum_{j=1}^n \Pr\{x_k = i, x_{k-1} = j, y_1, \dots, y_{k-1}, y_k\} \\ &= \sum_{j=1}^n \Pr\{y_k | x_k = i, x_{k-1} = j, y_1, \dots, y_{k-1}\} \Pr\{x_k = i | x_{k-1} = j, y_1, \dots, y_{k-1}\} \Pr\{x_{k-1} = j, y_1, \dots, y_{k-1}\} \\ &= \Pr\{y_k | x_k = i\} \sum_{j=1}^n \Pr\{x_k = i | x_{k-1} = j\} \Pr\{x_{k-1} = j, y_1, \dots, y_{k-1}\} \\ &= C_k(i) \sum_{j=1}^n a_{j,i}(k) \alpha_{k-1}(j), \end{aligned} \quad (6)$$

where

$$C_k(i) = \Pr\{y_k | x_k = i\}, \quad (7)$$

are conditional observation likelihoods (and called observation probability distributions in [8]). The conditional observation likelihoods are assumed to be known since they can be calculated entirely from the measurements (3).

The  $\alpha_k(i)$  are filtered probability distributions and are often termed forward probabilities. The filtered *indices* (under temporary Assumption 2) are obtained as the largest or maximum *a posteriori* (MAP) estimate [8] - [9],

$$i = \arg \max_{1 \leq i \leq n} \alpha_k(i). \quad (8)$$

Returning to Assumption 1, the filtered *values* are

$$\hat{x}_k = q_i. \quad (9)$$

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"An approximate answer to the right problem is worth a good deal more than an exact answer to an approximate problem." *John Wilder Tukey*

Thus, the HMM filter for the signal model (1) – (3) is parameterised by a transition probability matrix (5), conditional observation likelihoods (7), together with *a posteriori* state probabilities (6), and the filtered values arise as the most likely estimate (9).

### 11.3.2 HMM Fixed-Interval Smoother

The smoothing objective is to find optimal estimates  $\hat{x}_{k/T} \in \{1, \dots, n\}$  of the hidden states  $x_k$ , given measurements  $y_k$  over the entire fixed interval  $k \in [1, \dots, T]$ . The smoothed estimates are found using the well-known forward-backward algorithm [8], which requires the forward probabilities that were derived in the previous section, together with the backward probabilities of observations  $y_{k+1}$  to  $y_T$  occurring given that  $x_k = i$ . From the approach of [8], the backward probabilities are defined as

$$\beta_k(i) = \Pr\{y_{k+1}, \dots, y_T \mid x_k = i\}. \quad (10)$$

A recursive expression for  $\beta_k(i)$  is derived below. Applying the law of total probability, the joint probabilities formula in (10) and simplifying using the Markov property for observations yields

$$\begin{aligned} \beta_k(i) &= \sum_{j=1}^n \Pr\{y_{k+1}, y_{k+2}, \dots, y_T, x_{k+1} = j \mid x_k = i\} \\ &= \sum_{j=1}^n \Pr\{y_{k+2}, \dots, y_T \mid y_{k+1}, x_k = i, x_{k+1} = j\} \Pr\{y_{k+1}, x_{k+1} \mid x_k = i\} \\ &= \sum_{j=1}^n \Pr\{y_{k+2}, \dots, y_T \mid y_{k+1}, x_k = i, x_{k+1} = j\} \Pr\{y_{k+1} \mid x_{k+1} = j, x_k = i\} \Pr\{x_{k+1} = j \mid x_k = i\} \\ &= \sum_{j=1}^n \Pr\{y_{k+2}, \dots, y_T \mid x_{k+1} = j\} \Pr\{y_{k+1} \mid x_{k+1} = j\} \Pr\{x_{k+1} = j \mid x_k = i\} \\ &= \sum_{j=1}^n \beta_{k+1}(j) C_{k+1}(j) a_{j,i}(k+1). \end{aligned} \quad (11)$$

Finally, the smoothed posterior probability distributions are calculated as

$$\begin{aligned} \gamma_k(i) &= \sum_{j=1}^n \Pr\{x_k = i \mid y_1, \dots, y_T\} \\ &= \sum_{j=1}^n \frac{\Pr\{x_k = i \mid y_1, \dots, y_t, y_{t+1}, \dots, y_T\}}{\Pr(y_1, \dots, y_T)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n \frac{\Pr\{x_k = i \mid y_{k+1}, \dots, y_t \mid x_t = i\} \Pr\{x_t = i, y_1, \dots, y_T\}}{\Pr(y_1, \dots, y_T)} \\
&= \frac{\beta_k(i) \alpha_k(i)}{P(y_1, \dots, y_T)}.
\end{aligned} \tag{12}$$

The normalisation term within (12) can be obtained as  $\Pr\{y_1, \dots, y_k\} = \sum_{i=1}^n \beta_k(i) \alpha_k(i)$  to ensure that  $\sum_{i=1}^n \gamma_k(i) = 1$ . The smoothed *indices* (under temporary Assumption 2) are obtained as the largest or maximum *a posteriori* (MAP) estimate [8] - [9],

$$i = \arg \max_{1 \leq i \leq n} \gamma_k(i). \tag{13}$$

Returning to Assumption 1, the smoothed *values* are

$$\hat{x}_{k/N} = q_i. \tag{14}$$

Thus, the HMM smoother for the signal model (1) – (3) is parameterised by the forward, backward and smoothed probabilities, (6), (11) and (12), respectively. The smoothed values arise as the MAP estimate (14).

The HMM filter and smoother designs require estimates for  $A$  and  $C$ , which can be obtained in the following.

### 11.3.3 Transition Probability Matrix Estimation

Recall that a transition probability matrix describes the transitions of a Markov chain, and under Assumption 1 is defined as  $A = \{a_{i,j}\} \in \mathbb{R}^{n \times n}$ ,  $a_{i,j} = \Pr\{x_{k+1} = q_i \mid x_k = q_j\}$ . The  $a_{i,j}$  are the probabilities of transitioning from the current (column) state to the next (row) state. Thus, the columns of  $A$  represent families of probability distributions.

Suppose hypothetically that a state in a Markov chain is absorbing, namely, if once the state is entered, it is impossible to leave. In this case, the column of  $A$  corresponding to the state has a one on the main diagonal and zeros elsewhere. Thus, if all the states are absorbing,  $A$  would be an identity matrix. Next consider a more general case where states in a Markov chain change slowly from their neighbouring states. In this case, the components of  $A$  would be distributed about the main diagonal.

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“There may be babblers, wholly ignorant of mathematics, who dare to condemn my hypothesis... I value them not and scorn their unfounded judgement.” *Nicolaus Copernicus*

Now suppose that the probability distributions about the main diagonal of  $A$  are Gaussian. Similarly to the approach in [9], assume that  $a_{i,j} \sim \mathcal{N}(q_j - \phi q_i, \sigma_x^2)$ , where  $\phi$  is an autoregressive-order-1 coefficient that has been estimated for the process  $x_k$ , the  $q_i$  are linearly spaced over the range of  $x_k$  and  $\sigma_x^2$  is its sample variance. It follows that the components of  $A$  can be estimated as

$$a_{i,j} = (\sqrt{2\pi}\sigma_x)^{-1} \exp(-(q_j - \phi q_i)^2 / (2\sigma_x^2)). \quad (15)$$

The construction (15) results in each column of  $A$  being a vector of Gaussian probability densities.

### 11.3.4 Conditional Observation Likelihood Estimation

Suppose that the measurement noise within (3) is a zero-mean, Gaussian process, i.e.,  $v_k \sim \mathcal{N}(0, \sigma_v^2)$ . Following the approach of [9], it may be assumed that  $C_k(i) \sim \mathcal{N}(y_k - q_i, \sigma_v^2)$ . It follows that the components of  $C$  can be estimated as

$$C_k(i) = (\sqrt{2\pi}\sigma_v)^{-1} \exp(-(y_k - q_i)^2 / (2\sigma_v^2)). \quad (16)$$

Similarly, the construction (16) results in each column of  $C$  being a vector of Gaussian probability densities.

**Example 2:** A 6-second utterance of “a e i o u” was sampled at 44.1 kHz and recorded in an Mpeg-4 audio format to serve as an example speech signal  $x_k \in \mathbb{R}$ . Noisy measurements  $y_k$  were then generated by adding Gaussian, white noise realisations to the speech signal. Minimum-variance filters and smoothers were designed assuming a signal model  $x_{k+1} = \phi x_k + w_k$  with parameters  $\phi = E\{y_{k+1}y_k^T\}(E\{y_k y_k^T\} - \sigma_y^2)^{-1}$  and  $\sigma_w^2 = E\{y_k y_k^T\} - R - A(E\{y_k y_k^T\} - R)A^T$  estimated from the (noisy) measurements. The root-mean-square (RMS) error performance of the minimum-variance filters and smoothers are indicated by the dotted lines of Fig. 1(a) and 12(b), respectively. The measurements were also quantised into  $n = 32$  levels. The performance of HMM filters and smoothers, with (15) – (16) estimated from the measurements, are indicated by the dashed lines of Fig. 1(a) and 1(b), respectively. The performance of HMM filters and smoothers, with (15) – (16) estimated from noiseless speech signals, are indicated by the solid lines of Fig. 1(a) and 1(b), respectively. It can be seen that the HMM filters/smoothers can outperform the optimal minimum-variance solutions at low SNR, provided that noiseless “training” data is available for parameter estimation. The figures also demonstrate that signal quantisation degrades fidelity and leads to degraded high-SNR performance.

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“A life spent making mistakes is not only more honourable, but more useful than a life spent doing nothing.” *George Bernard Shaw*



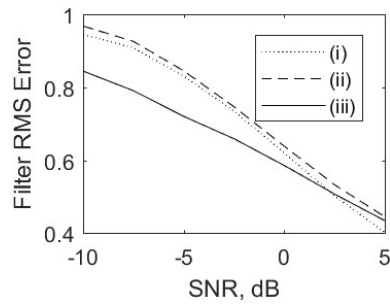


Fig. 1(a) RMSE versus SNR for: (i) Minimum-variance filter; (ii) HMM filter trained on noisy measurements; and (iii) HMM filter trained on noiseless signals.

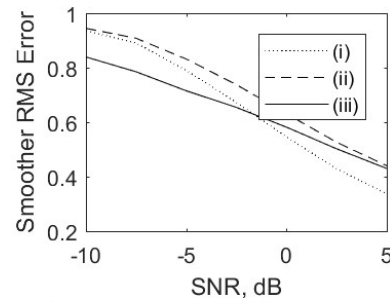


Fig. 1(b) RMSE versus SNR for: (i) Minimum-variance smoother; (ii) HMM smoother trained on noisy measurements; and (iii) HMM smoother trained on noiseless signals.

## 11.4 Minimum-Variance-HMM Filtering and Smoothing

### 11.4.1 Motivation

Optimal filters and smoothers [1] - [3], [12] minimise the error variance by exploiting knowledge of the signal generating and measurement noise processes. Many tracking problems involve repetitive trajectories. Within surface mines, for example, trucks repeatedly drive up and down the same haul roads. On stockpiles, dozers travel up and down similar inclines/declines. At export terminals, vehicles are driven along similar trajectories. This section is concerned with improving minimum-variance filter and smoother performance by additionally exploiting knowledge about such repetitive trajectories.

In optimal Kalman filtering and smoothing applications [1] - [3], [12], it is routinely assumed that one or more state sequences are generated by a random walk or an autoregressive process. If the signal generation process has a low-pass spectrum then the optimal filter and smoother will also have low-pass spectra. That is, common Kalman filter parameterisations do not exploit knowledge about the random variables' probability distributions. This shortcoming has motivated the development of the previously-discussed HMM filters and smoothers.

Kalman and HMM filters/smoothers have different strengths and weaknesses. The optimal Kalman filter and smoother solutions minimise the variance of the estimation error. HMM filters/smoothers exploit knowledge about the random variables' probability distributions and are optimal in a Bayesian sense but do not explicitly minimise the mean-square error (MSE). They model systems as Markov

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"There's nothing quite as frightening as someone who knows they are right." *Michael Faraday*

chains which have mutually exclusive states. Consequently, HMM filter outputs are discretised. Kalman filters employ predictions that involve linear combinations of previous states. Thus, Kalman filters are less suited to problems possessing mutually exclusive states and their outputs are not constrained by discretisation boundaries.

Improved estimation performance is sought by exploiting *both* transition probability knowledge and minimum-error-variance optimality. This entails finding a state-space realisation of a stochastic process whose output is equivalent to that of an HMM. The desired stochastic model may then be used within filter/smoothing designs to attain MSE performance benefits.

#### 11.4.2 Prior Literature

There are two common solution approaches for including transition probability knowledge within filter/smoothing designs. First, implementing coupled Kalman and HMM filters [14] - [22]. Second, parameterising a single minimum-variance filter for recovery of Markov chain states [15], [23] - [26].

An extended Kalman filter is coupled with an HMM filter in [14] - [15] for demodulating differential phase shift keyed and frequency modulated signals. A coupled HMM filter and Kalman filter can be used for interference suppression [16]. The merging of a HMM and a Kalman filter for tracking space vessel CO<sub>2</sub> concentrations is described in [17]. Time-domain and frequency-domain HMMs can be used to identify model parameters prior to filtering and smoothing of noisy speech [18] - [20]. In a position tracking application [21], an HMM is used to estimate a control input for a Kalman filter. In an image target tracking problem [22], an HMM is used to estimate position coordinates which then serve as a measurement input for a Kalman filter.

Rather than coupling two filters, knowledge of a signal's probability distributions can be used in the parameterisation of a single minimum-variance filter. This approach has been considered previously in [15], [23] - [26]. In [15], a system's input is termed a semi-Martingale increment. A single-output, time-invariant, linear state model, which has the same second-order statistics as a hidden Markov chain (HMC), is described without proofs in [15]. A state-space system can be transformed into an innovations or prediction error representation [24] - [26]. It is assumed in [15], [25] that the transition probability matrix and the output matrix are known from which an input process covariance is calculated. These parameters are used to calculate an algebraic Riccati equation solution for which convergence is established in [25]. A subspace identification algorithm arises from a factorisation of the singular value decomposition of an estimated linear prediction matrix in [24]. This yields least-squares estimates of the transition probability and

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"Torture the data and it will confess to anything." *Ronald Coase, winner of the 1991 Nobel Prize in Economics*

gain matrices, without having to explicitly estimate an input covariance and solve a Riccati equation. A rigorous understanding of Markov model subspace identification is developed in [26]. Although convergence and consistency results are developed, it is concluded in [26] that the problem of finding a stochastic matrix solution for a transition probability matrix remained unsolved.

Many other techniques have been reported for identifying model parameters – see [27] - [39] and the references therein. The statistical properties of quantised variables is surveyed in [27] – [29] and the impact of quantisation on moment or likelihood based approaches to system identification is studied in [31] - [32]. Candidate parameter estimation techniques include least-squares, EM algorithms, sub-space identification and convex optimisation. A closed-form least-squares estimate of a transition probability matrix is stated in [33]. EM algorithms are described in [8], [35]. Sub-space identification methods, which are also known as spectral algorithms, are surveyed in [24] - [26], [36] - [39]. These methods find approximate factorisations of estimated covariances between past and future observations. A spectral algorithm for learning discrete-observation HMMs is proposed in [37], which does not require estimating the transition and observation matrices. The spectral algorithm of [37] is generalised for HMMs that possess kernel structures in [38]. The assumptions within [37] are relaxed for which reduced-rank HMMs are developed in [39].

### 11.4.3 Signal Models

#### 11.4.3.1 Hidden Markov Model

Consider again a time-homogenous Markov chain with  $n$  states,  $X_k$ ,  $k \geq 0$ , which takes integer values, without loss of generality, in a finite set  $\{1, 2, \dots, n\} \subset \mathbb{R}$ . Denote  $\Pi = \text{diag}(\pi_1, \dots, \pi_n)$ , where  $\pi = [\pi_1, \pi_2, \dots, \pi_n]^T \in \mathbb{R}^n$ ,  $\pi_i = \Pr\{X_k = i\}$ , is the unique stationary probability distribution of the states which satisfies

$$\pi = A\pi, \quad (17)$$

in which  $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$  is a column-stochastic transition probability matrix for  $X_k$ , i.e.,  $a_{ij} = \Pr(X_{k+1} = i | X_k = j)$ . Let  $Y_k$ ,  $k > 0$  be a HMC, dependent on  $X_k$ , taking on values in a set  $\mathcal{Y} \subseteq \mathbb{R}^L$  with probability distribution

$$E\{Y_k\} = M\pi, \quad (18)$$

where  $M \in \mathbb{R}^{L \times n}$  is an observation probability matrix. A brief derivation of unconditional second-order moments of the HMC, namely,  $E\{Y_{k+\tau} Y_k^T\}$  and  $E\{Y_k Y_k^T\}$ , is provided below.

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“I keep saying that the sexy job in the next 10 years will be statisticians, and I’m not kidding.” *Hal Varian, Chief Economist, Google*

Following the approach in [23], assume there exist bounded conditional first and second order moments  $m_i \in \mathbb{R}^L$  and  $\Sigma_i \in \mathbb{R}^{L \times L}$  defined by  $m_i = E\{Y_k | X_k = i\}$  and  $\Sigma_i = E\{Y_k Y_k^T | X_k = i\}$ , respectively. It is observed that  $E\{Y_k\} = \sum_{i=1}^n \pi_i m_i = M\pi$ , where  $M = [m_1 \ m_2 \ \dots \ m_n] \in \mathbb{R}^{L \times n}$ . Denote  $\Phi(\tau) = \{\Phi_{ij}(\tau)\} \in \mathbb{R}^{n \times n}$ ,  $\Phi_{ij}(\tau) = \Pr\{X_{k+\tau}=i | X_k=j\}$  for  $\tau > 0$ , which is consistent with  $\Phi(0) = I_n$  and  $\Phi(\tau) = A^\tau$ . Assume that  $Y_k$  satisfy the conditional independence property

$$\Pr\{Y_0, \dots, Y_k | X_0, \dots, X_t\} = \prod_{\tau=0}^k \Pr\{Y_\tau | X_\tau\}, \quad (19)$$

Without loss of generality, consider a simpler case in which the HMC means  $\mu = M\pi$  are zero for all  $k \geq 0$ . Condition (19) ensures,  $E\{Y_{k+\tau} Y_k^T | X_{k+\tau} = j, X_k = i\} = m_i m_j^T$  when  $\tau \neq 0$ . Now the (unconditional) second order moments of the HMC  $Y_t$  can be determined. For  $\tau > 0$ ,  $E\{Y_{k+\tau} Y_k^T\} = \int_{y_1 y_2} y_1 y_2^T \Pr_{Y_{k+\tau}, Y_k} \{y_1, y_2\} dy_1 dy_2 = \int_{y_1 y_2} y_1 y_2^T \sum_{i,j=1}^n \Pr_{Y_{k+\tau}, Y_k | X_{k+\tau}=i, X_k=j} \{y_1, y_2\} \Pr(X_k = j, X_{k+\tau} = i) dy = \sum_{i,j=1}^n m_i m_j^T \pi_j \Phi_{i,j}(\tau) = M\Phi(\tau)\Pi M^T$ . Also,  $E\{Y_k Y_k^T\} = \int_{y_1 y_2} y_1 y_2^T \Pr_{Y_k} \{y\} dy = \int_{y_1 y_2} y_1 y_2^T \sum_{i=1}^n \Pr_{Y_k | X_k=i} \{y\} \Pr\{X_k = i\} dy = \sum_{i=1}^n (V_i + m_i m_i^T) \pi_i = \sum_{i=1}^n V_i \pi_i + M\Pi M^T$ , where  $V_i = \text{cov}(Y_k | X_k = i, Y_k | X_k = i) > 0$  denotes the conditional observation covariance given  $X_k = i$ .

### 11.4.3.2 State-Space Model

Now consider the linear, time-invariant, state-space model

$$x_{k+1} = Fx_k + Gu_k \quad (20)$$

$$y_k = Hx_k + Jw_k, \quad (21)$$

where  $x_k, u_k \in \mathbb{R}^n$ ,  $y_k, w_k \in \mathbb{R}^L$ ,  $F, G \in \mathbb{R}^{n \times n}$ ,  $H \in \mathbb{R}^{L \times n}$  is known as an output mapping and  $J \in \mathbb{R}^{L \times L}$ . The exogenous input sequences  $u_k, w_k$  are zero-mean, uncorrelated, white processes with unit covariances, i.e.,  $E\{u_k u_s^T\} = \delta_{s,k} I_N$ ,  $E\{w_k w_s^T\} = \delta_{s,k} I_M$ ,  $E\{u_k w_s^T\} = 0$ , where  $\delta_{s,k}$  denotes the Kronecker delta

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“For every fact there is an infinity of hypotheses.” *Robert Pirsig, Zen and the Art of Motorcycle Maintenance*

function and  $I_n \in \mathbb{R}^{n \times n}$ ,  $I_M \in \mathbb{R}^{M \times M}$  are identity matrices. Derivations of predictors, filters and smoothers based on the state-space model (20) – (21) are detailed in Chapters 5 and 7.

### 11.4.3.3 Equivalent Markov Model

In order to design a filter and smoother that exploit transition probability knowledge and attain minimum-variance optimality, a state-space model that possesses a transition probability matrix and vector observations is developed. Following the approach in [23], it is shown below that the state space system (20) – (21) can be parameterised so that its output covariances match those for the HMC. Let  $V_i = \text{cov}(Y_k|X_k = i, Y_k|X_k = i) > 0$  denote the conditional observation covariance given  $X_k = i$ .

**Lemma 1 [23]:** In respect of the above HMM and the state-space system (20) – (21), suppose that 1)  $A$  and  $\Pi$  are known, then the parameterisation:

2)  $H = M$ ;

3)  $F = A - \Pi \mathbf{1}^T$ , where  $\mathbf{1} = [1, \dots, 1]^T$ ;

4)  $P = \Pi > 0$ ;

5)  $G$  is any non-singular matrix satisfying  $GG^T = \Pi - F\Pi F^T > 0$ ;

$J$  is any non-singular matrix satisfying  $JJ^T = \sum_{i=1}^n \pi_i V_i > 0$ , results in:

$$(a) E\{Y_k Y_k^T\} = E\{y_k y_k^T\};$$

$$(b) E\{Y_{k+\tau} Y_k^T\} = E\{y_{k+\tau} y_k^T\};$$

for all  $k$  and  $\tau$ .

**Proof:** (a) It follows from (21) that  $E\{y_k y_k^T\} = HPH^T + JJ^T$ , which together with conditions 1) and 5) yield the result.

(b) It is assumed that the initial mean  $E\{Z_0\} = 0$  and covariance  $P = E\{x_0 x_0^T\} = E\{x_k x_k^T\}$  is the solution to the Lyapunov equation  $P = FPF^T + GG^T$ . Note that for  $\tau > 0$ ,

$$x_{k+\tau} = \psi(\tau)x_k + \sum_{s=t}^{k+\tau-1} \psi(t+\tau-s-1) Gu_s, \quad (22)$$

where  $\psi(\tau) = F^\tau$  for  $\tau \geq 0$ . So  $E\{x_k\} = 0$  for all  $k \geq 0$ . From (22) and the properties of the process  $u_k$  (i.e.,  $E\{u_k x_s^T\} = 0$  for all  $s \leq k$ ) it holds that for  $\tau > 0$ ,

$$E\{x_{k+\tau} x_k^T\} = \psi(\tau)E\{x_k x_k^T\} + \sum_{s=k}^{k+\tau-1} \psi(k+\tau-s-1)GE\{u_s x_k^T\} = \psi(\tau)P, \quad (23)$$

---

“The great tragedy of science – the slaying of a beautiful hypothesis by an ugly fact.” Thomas Henry Huxley

and for  $\tau < 0$ ,  $E\{x_{k+\tau}x_k^T\} = P^T\psi^T(-\tau)$ . It follows from (21) and (23) for  $\tau > 0$  that

$$E\{y_{k+\tau}y_k^T\} = H\psi(\tau)PH^T. \quad (24)$$

Thus, from Conditions 2) and 4), and (21), it needs to be shown for  $\tau > 0$  that

$$M\Phi(\tau)\Pi M^T = H\psi(\tau)PH^T. \quad (25)$$

For the initialisation step of an inductive argument, suppose (25) holds for  $\tau = 1$ , namely,  $H\psi(1)PH^T = M\Pi M^T = M\Phi(1)\Pi M^T$ , since it is assumed that  $M\pi = 0$ . For the inductive step, suppose (25) holds for some  $\tau > 0$  and consider  $H\psi(\tau+1)PH^T = MF\Phi(\tau)\Pi M^T = MA\Phi(\tau)\Pi M^T = M\Phi(\tau+1)\Pi M^T$ .

The probability transition matrix  $A$  possesses a maximal eigenvalue of 1 (see [41]). Condition 3) of Lemma 1 shifts the unstable eigenvalue to the origin and leaves the other eigenvalues unchanged. Thus, Lemma 1 specifies the matrices associated with a *stable* linear state space model so that the second-order moments of its output match those of a specified HMM. The input matrix  $G$  can be selected so that the state covariance is diagonal (see also [23], [25]). That is, setting  $P = \Pi > 0$ , and noting that  $F$  is stable, a non-singular  $G$  can be found by Cholesky factorisation of  $\Pi - F\Pi F^T$ . This parameterisation is illustrated by the following example.

**Example 3:** Consider  $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ , for which  $\pi = [0.6 \quad 0.4]^T$ . A stable state matrix is found from  $F = A - \pi\pi^T = \begin{bmatrix} 0.1999 & -0.3 \\ -0.2 & 0.2999 \end{bmatrix}$ . By setting  $P = \Pi = \text{diag}(0.6, 0.4)$ , an input covariance is obtained as  $GG^T = P - FPF^T$  and a Cholesky factorisation yields  $G = \begin{bmatrix} 0.7348 & 0 \\ 0.0816 & 0.5773 \end{bmatrix}$ . By construction, the solution of  $P = FPF^T + GG^T$  is  $P = \Pi$ .

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“You never change things by fighting the existing reality. To change something, build a new model that makes the existing model obsolete.” *Richard Buckminster Fuller*

## 11.4.4 Filter and Smoother Design Procedure

### 11.4.4.1 Problem Assumptions

The usual problem in HMC estimation is to determine the *a posteriori* probabilities of the states at each time, given the observations (in either a filtering or a smoothing context). Problems such as this can be solved using HMM filters and smoothers such as [8] - [10] discussed in Section 11.3, in which it is assumed that the underlying model is linear and the noises are Gaussian.

A linear state space model is described above that motivates the application of the optimal filter and optimal smoother to discretised signals - which do not rely on Gaussian noise assumptions. Let  $A_n$ ,  $F_n$ ,  $H_n$  and  $G_n$ , respectively denote  $A$ ,  $F$ ,  $H$  and  $G$  at  $n$  discretisation levels. The following is assumed: (1) observations  $y_k = Y_k$  are available; (2)  $H_n$  and  $n$  are selected by the designer; (3) the measurement noise covariance  $JJ^T$  can be estimated from the observations during periods when the signal is absent; (4)  $F_n$  and  $G_n$  can be identified; and (5) the states  $x_k$  can be estimated by a linear filter or a smoother that operate on  $Y_k$ .

### 11.4.4.2 Identified Parameters

Suppose that observations  $Y_k$ ,  $k \in [1, N]$ , are discretised into  $n$  levels to yield initial estimates  $\hat{x}_k = [\hat{x}_{1,k}, \dots, \hat{x}_{n,k}]^T = q_n(Y_k)$  of states  $x_k$ , where  $q_n: \mathbb{R} \rightarrow \mathbb{R}^n$  denotes a discretisation function that produces

$$\hat{x}_{j,k} = \begin{cases} 1 & \text{if } h_j - \Delta/2 \leq Y_k < h_j + \Delta/2 \\ 0 & \text{otherwise} \end{cases}, \quad (26)$$

in which  $h_1 \dots h_n \in \mathbb{R}$  are discretisation level centres spaced  $\Delta$  apart. Let  $H_n = [h_1, \dots, h_n] \in \mathbb{R}^{1 \times n}$  and denote its pseudo inverse by  $H_n^\dagger$ . The initial state estimates may be written as  $\hat{x}_k = H^\dagger Y_k = x_k + H^\dagger J w_k$ . Since  $x_i$  and  $w_k$  are independent, it follows that

$$\hat{x}_k \hat{x}_k^T = x_k x_k^T + H_n^\dagger J J^T H_n^{\dagger T}. \quad (27)$$

Standard least-squares estimates of  $A_n$  and  $G_n G_n^T$  can be calculated from [23], [25], [26] and [33] as

$$A_n = E\{\hat{x}_{k+1} \hat{x}_k^T\} E\{\hat{x}_k \hat{x}_k^T\}^{-1}, \quad (28)$$

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“Without imperfection, you or I would not exist.” *Stephen William Hawking*

$$G_n G_n^T = E\{\hat{x}_{k+1} \hat{x}_k^T\} - A_n (E\{\hat{x}_k \hat{x}_k^T\} A_n^T). \quad (29)$$

Alternatively,  $A_n$  may be constructed as columns of conditional probability distributions, *i.e.*,

$$A_n = \Pr(\hat{x}_{i,k+1} = 1 | \hat{x}_{j,k} = 1) = \frac{\sum_{k=1}^{N-1} \hat{x}_{i,k+1} \hat{x}_{j,k}^T}{\sum_{k=1}^{N-1} \hat{x}_{j,k} \hat{x}_{j,k}^T}$$

and

$$A_n = \sum_{k=1}^{N-1} \begin{bmatrix} \hat{x}_{1,k+1} \\ \vdots \\ \hat{x}_{i,k+1} \\ \vdots \\ \hat{x}_{n,k+1} \end{bmatrix} \begin{bmatrix} \hat{x}_{1,k} \\ \vdots \\ \hat{x}_{i,k} \\ \vdots \\ \hat{x}_{n,k} \end{bmatrix}^T \left( \sum_{k=1}^{N-1} \begin{bmatrix} \hat{x}_{1,k} \\ \vdots \\ \hat{x}_{j,k} \\ \vdots \\ \hat{x}_{n,k} \end{bmatrix} \begin{bmatrix} \hat{x}_{1,k} \\ \vdots \\ \hat{x}_{j,k} \\ \vdots \\ \hat{x}_{n,k} \end{bmatrix}^T \right)^{-1} = E\{\hat{x}_{k+1} \hat{x}_k^T\} (E\{\hat{x}_k \hat{x}_k^T\})^{-1},$$

which is identical to (28). Thus, the least-squares estimate (28) is a stochastic transition probability matrix. The ensuing filter and smoother require that (29) is positive definite which was established by the author of [23] as set out below.

**Lemma 2:** Suppose that  $F_n$  is a stable matrix obtained from  $A_N$  using Condition (3) of Lemma 1 and  $P_N$  is a symmetric positive definite matrix, then the matrices (a)  $Q_n = P_n - F_n P_n F_n^T$  and (b) (29) are positive definite.

**Proof:** (a) Since  $F_n$  is stable, it has an eigenvalue decomposition  $F_n = U \Lambda U^{-1}$  where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  is generally a complex diagonal matrix with  $|\lambda_i| < 1$ ,  $i = 1 \dots n$ . Let  $F_n^H$  denote the Hermitian conjugate of  $F_n$ . We have  $Q_n = P_n - F_n P_n F_n^T = P_n - U \Lambda U^{-1} P_n (U^H)^{-1} \Lambda^H U^H$ , *i.e.*,

$$\tilde{Q}_n = \tilde{P}_n - \Lambda \tilde{P}_n \Lambda^H, \quad (30)$$

where  $\tilde{Q}_n = U^{-1} Q_n (U^H)^{-1}$  and  $\tilde{P}_n = U^{-1} P_n (U^H)^{-1}$ . Note that  $\tilde{P}_n$  is positive if and only if  $P$  is, and  $\tilde{Q}_n$  is positive definite if and only if  $Q$  is. Let  $\text{Vec}(\cdot)$  denote the columns of a matrix stacked into a vector. Equation (30) implies  $\text{Vec}(\tilde{Q}) = (I - \Lambda^H \otimes \Lambda) \text{Vec}(\tilde{P}_n) = ((I - \Lambda^H) \otimes (I - \Lambda)) \text{Vec}(\tilde{P}_n)$ . Now a Hermitian matrix  $W$  is positive definite if and only if  $x^H W x > 0$  for all nonzero vectors  $x$ . This is equivalent to the condition that  $x^T \otimes x^H \text{Vec}(W) > 0$  for all nonzero vectors  $x$ . Now let  $x \in \mathbb{C}^n$  be nonzero and consider

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"Failure is only the opportunity to begin again more intelligently." Henry Ford



$$\begin{aligned}
x^T \otimes x^H \text{Vec}(\tilde{Q}_n) &= (x^T \otimes x^H)((I - \Lambda^H) \otimes (I - \Lambda)) \text{Vec}(\tilde{P}_n) \\
&= ((x^T (I - \Lambda^H)) \otimes (x^H (I - \Lambda))) \text{Vec}(\tilde{P}_n) \\
&= (y^T \otimes y^H) \text{Vec}(\tilde{P}_n),
\end{aligned} \tag{31}$$

where  $y = (I - \Lambda^H)x$  is an invertible mapping since  $I - \Lambda^H$  is non-singular. It can be seen from (31) that since  $\tilde{P}_n$  is positive definite,  $\tilde{Q}_n$  is positive definite.

(b) The result follows by considering  $P_n = \hat{x}_k \hat{x}_k^T - H_n^# J J^T H_n^{\#T} > 0$  within (a).

#### 11.4.4.3 Model Order Design

The performance of the filter and smoother described below depend on the number of discretisations. Coarse discretisation can result in a loss of (recovered) signal fidelity. If the discretisation is too fine the empirically derived transition matrices may be rank deficient which can affect observability and reachability.

Consider the observability matrix  $O(F_n, H_n) = \begin{bmatrix} H_n \\ H_n F_n \\ \vdots \\ H_n F_n^{n-1} \end{bmatrix}$  and the reachability

matrix  $O(F_n^T, H_n^T)^T = [G_n \quad F_n G_n \quad \cdots \quad F_n^{n-1} G_n]$ . If  $O(F_n^T, H_n^T)^T$  is rank deficient not all modes of the linear system will be excited. If  $O(F_n, H_n)$  is not of full rank then not all the states can be recovered from the system's output.

The statistical properties of quantised variables is surveyed in [27] and stochastic observability for HMM filters is addressed in [28]. However, the studies on quantisation and system identification [27] - [32], and observability [28] - [40] do not provide guidance on the choice for  $n$ . Consequently, a model-order reduction decision process is developed in the following.

Checking if  $O(F_n, H_n)$  is of rank  $n$  is equivalent to checking if the function  $f(n, O(F_n, H_n)) = n - \text{rank}(O(F_n, H_n))$  is positive. It is noted under simplifying conditions below that this function increases monotonically with  $n$ .

**Lemma 3:** Suppose that (i)  $F_n$  is a diagonal matrix so that  $f(n, O(F_n, H_n)) = 0$ . Then  $f(n+i, O(F_{n+i}, H_{n+i})) \geq f(n+i+1, O(F_{n+i+1}, H_{n+i+1}))$  for integer  $i \geq 0$ .

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“Technological progress is like an axe in the hands of a pathological criminal.” *Albert Einstein*

**Proof:** Since the diagonal of the transition probability matrix possesses zeros for integer  $i \geq 0$ , it follows that  $\text{rank}(F_{n+i+1}) = \text{rank}(F_{n+i})$ , from which the claim follows.

It can similarly be shown that  $f(n, O(F_n^T, G_n^T)^T) = N - \text{rank}(O(F_n^T, G_n^T)^T)$  increases monotonically with  $N$ . Therefore, it is advocated that the model order is reduced whenever  $(F_n, H_n)$  is unobservable or  $(F_n, G_n)$  is unreachable, i.e.,

$$\begin{aligned} n \rightarrow n-1 \text{ if } f(n, O(F_n, H_n)) > 0 \\ \text{or } f(n, O(F_n^T, H_n^T)^T) > 0. \end{aligned} \quad (32)$$

By deflating the model to enforce observability and reachability, an identified linear system will have a minimal realisation (see Thm. 2.4-6 of [43]).

#### 11.4.4.4 Filter and Smoother Recursions

Optimal causal estimates of the Markov chain states can be obtained by operating a minimum-variance filter [1] on the HMC measurement data  $Y_k$  at time  $k$ . Let  $\hat{x}_{k/k}$  and  $\hat{x}_{k+1/k}$  respectively denote filtered and predicted state estimates given data at time  $k$ . These estimates are calculated as

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + \bar{P}H_n^T\Omega^{-1}(Y_k - H_n\hat{x}_{k/k-1} - H_n\pi), \quad (33)$$

$$\hat{x}_{k+1/k} = F_n\hat{x}_{k/k}. \quad (34)$$

where  $K = F_n\bar{P}H_n^T\Omega^{-1}$  is the predictor gain,  $\Omega = H_n\bar{P}H_n^T + JJ^T$ , in which  $\bar{P} \in \mathbb{R}^{n \times n}$  is the solution of the ARE  $\bar{P} = F_n\bar{P}F_n^T - K\Omega K^T + G_nG_n^T$ . The inclusion of  $H_n\pi$  within (33) ensures that the filtered state estimates are zero-mean. The filtered output is given by  $\hat{y}_{k/k} = H_n\hat{x}_{k/k} + H_n\pi$  which retains the properties common to the optimum solution, namely, it is unbiased and minimises the variance of the output estimation error [12]. Noncausal estimates of the Markov chain states can be obtained by operating an optimal fixed-interval smoother [12] on the HMC measurements over  $k \in [1, N]$ . The smoothed output estimates,  $\hat{y}_{k/T}$ , can be obtained from (34),

$$\alpha_k = \Omega^{-1/2}(Y_k - H_n\hat{x}_{k/k-1} - H_n\pi), \quad (35)$$

$$\xi_{k-1} = (F_n^T - H_n^TK^T)\xi_k + H_n^T\Omega^{-1/2}\alpha_k, \quad \xi_N = 0, \quad (36)$$

$$\beta_k = -K^T\xi_k + \Omega^{-1/2}\alpha_k, \quad (37)$$

$$\hat{y}_{k/N} = Y_k - JJ^T\beta_k, \quad (38)$$

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“The factory is the machine that builds the machine.” *Elon Reeve Musk*

where  $\alpha_k, \beta_k \in \mathbb{R}$ ,  $\zeta_k \in \mathbb{R}^n$ . Similarly, the smoothed output estimates (38) are unbiased and minimise the variance of the output estimation error [12]. Next, conditions are stated for the filter (33) – (34) and smoother (35) – (38) to be asymptotically stable. Let  $\lambda_i(X)$  denote the  $i^{\text{th}}$  eigenvalue of  $X$ .

**Lemma 4:** *Under the conditions of Lemmas 1 – 2, additionally assume that  $(F_n, H_n)$  is observable,  $(F_n, G_n)$  is reachable and  $JJ^T > 0$ , then  $|\lambda_i(F_n - KH_n)| < 1, i = 1 \dots n$ .*

**Proof:** Condition 2) of Lemma 1 ensures that  $|\lambda_i(\hat{F}_n)| < 1, i = 1 \dots n$ , and Lemma 2 establishes that  $\hat{G}_n \hat{G}_n^T > 0$ , which together with the above additional assumptions satisfy the conditions of Thm. 2.1 of [42].

The estimation of discrete Markov chain states is usually the purview of optimal Bayesian solutions such as HMM filters/smothers and it may seem counter-intuitive that the linear filter and smoother can be employed for this task. It is verified below that the desired outputs can be recovered exactly when the measurement noise is negligible.

**Lemma 5:** (a)  $\lim_{J \rightarrow 0} \hat{y}_{k/k} = Y_k$ ; (b)  $\lim_{JJ^T \rightarrow 0} \hat{y}_{k/N} = Y_k$ .

**Proof:** (a) It follows from (33) that  $\hat{y}_{k/k} = (I - H_n \bar{P} H_n^T \Omega^{-1}) H_n \hat{Z}_{k/k-1} + H_n \bar{P} H_n^T \Omega^{-1} Y_k + (I - H_n \bar{P} H_n^T \Omega^{-1}) H_n \pi$  which together with  $\lim_{J \rightarrow 0} H_n \bar{P} H_n^T \Omega^{-1} = I$  yields  $\lim_{J \rightarrow 0} \hat{y}_{k/k} = Y_k$ .

(b) By inspection of (38),  $\lim_{JJ^T \rightarrow 0} \hat{y}_{k/N} = Y_k$ .

An understanding of why the filter becomes more precise at high SNR follows by recognising that the output estimator approaches a short circuit (and passes the measurements straight through). Thus filters and smoothers may be designed by discretising the observations (26), using the parameter estimates (28) – (29), enforcing stability (Lemma 1), observability and reachability (32).

The state correction (34), prediction (33) and output estimate respectively involve  $N$ ,  $N$  and 1 inner products of  $n$ -dimensional quantities. Hence, the total filter calculation cost over a length- $N$  interval is  $(2N+1)n$  inner products. The smoothed estimates are conveniently obtained by time-reversing the  $\beta_n$  (see, [12]), in which case  $(4N+1)n$  inner products are required over the interval. Thus, any observed

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“Progress imposes not only new possibilities for the future but new restrictions.” *Norbert Wiener*

performance benefits needs to be reconciled against the considerable increase in the number of calculations. In the event that the above calculation costs are prohibitive, it may be beneficial to design first-order filters and smoothers with  $A_1 = H_n F_n H_n^T$ , instead of estimating an  $A_n$  directly from (28).

### 11.4.5 Application

Improved position estimates are desired for tracking vehicles in cluttered mining environments. Raw Global Positioning System (GPS) measurements tend to be inaccurate whenever vehicle receivers operate during poor satellite geometry, within canyons and under obstructions. Multipath interference and radio spectrum congestion can also cause GPS signal degradation.

Additional sensors can be used to improve land-based navigation performance including inertial navigation systems (INS), cameras and laser scanners. Optimal filters and smoothers [1] - [3], [12] are typically used to mitigate sensor noise. For example, optimal filters are routinely employed in integrated GPS-INS positioning applications such as [13].

Vehicle position tracking technologies that improve productivity and safety are desired within mining and allied transport industries. Mining vehicle position tracking problems often exhibit the following two characteristics. First, position estimates are corrupted by noise. Second, the traverses are repetitive. At export terminals, vehicles are driven along similar trajectories. An example that is motivated by tracking dozers on a coal stockpile at an export terminal is described in the following.

Caterpillar D11 dozers are used to manage about 65 Mt of coal annually at the Port of Gladstone. When incoming coal is delivered by overhead conveyors, the dozers are required to push the coal onto stockpiles. Conversely, when coal is to be delivered to ships, the dozers are required to push the coal from the stockpiles to discharge feeders situated under the conveyors. The dozers repeatedly operate under the overhead conveyors and these overhead structures obscure the paths to GPS satellites and contribute to multipath interference. Techniques are therefore desired which improve the dozers' noisy GPS position estimates.

Zero-mean, unity-variance samples of northings, eastings and altitude positions reported by a Novatel OEMV-3-L1 RTK GPS receiver that travelled from the bottom to the top of a coal stockpile at Gladstone are shown in Fig. 2(a). A simulation study is now described in which measurement noise is added to the dozer position measurements. The objective of this study is to demonstrate that the filter (33) – (34) and smoother (13) – (16) can outperform conventional HMM and

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“In our struggle to understand the history of life, we must learn where to place the boundary between contingent and unpredictable events that occur but once and the more repeatable, law-like phenomenon that may pervade life's histories as generalities.” *Stephen Jay Gould*

minimum-variance filters and smoothers for recovering position estimates from the noisy measurements.

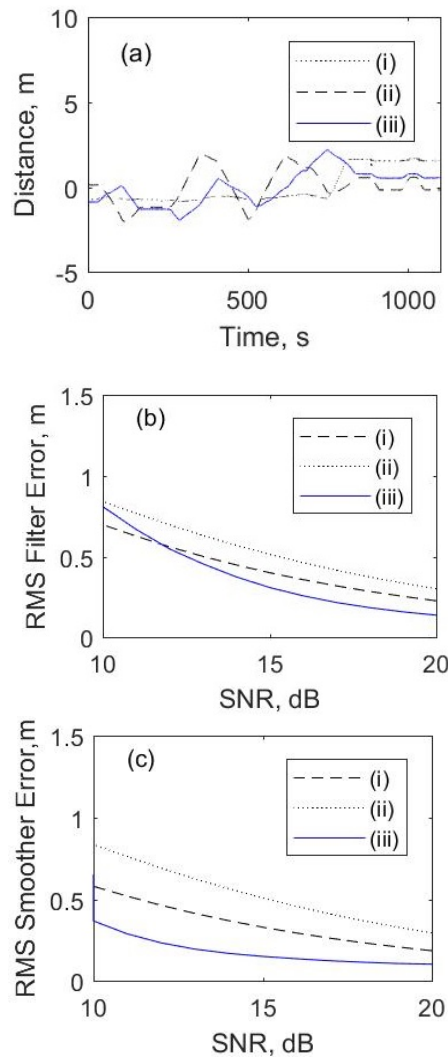


Fig. 2. Data for the position tracking example: (a) sample zero-mean unity-variance northings (i), eastings (ii) and altitude (ii) positions; (b) HMM filter (i), first-order minimum-variance filter (ii) and developed minimum-variance filter (iii) RMS position errors; (c) HMM smoother (i), first-order minimum-variance smoother (ii) and developed minimum-variance smoother (iii) RMS position errors.

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“New scientific ideas never spring from a communal body, however organized, but rather from the head of an individually inspired researcher who struggles with his problems in lonely thought and unites all his thought on one single point which is his whole world for the moment.” *Max Karl Ernst Ludwig Planck*

Independent Gaussian noise realisations were added to the GPS northings, eastings and altitudes. The northings, eastings and altitude measurements were quantised into  $N = 64$  levels. An HMM filter and a fixed-interval HMM smoother were implemented using the techniques described in Section 11.3. In particular, the forward, backward and smoothed probabilities were respectively calculated from (6), (11) and (12), assuming Gaussian probability distributions (15) – (16). The observed HMM filter and HMM smoother RMS error versus signal to noise ratio (SNR) are indicated by lines (i) of Fig. 2(b) and Fig. 2(c), respectively.

A first-order minimum-variance filter [1] - [3] and smoother [12] were also applied to the raw measurements. The RMS errors exhibited by the minimum-variance filter and smoother are indicated by lines (ii) of Fig. 2(b) and Fig. 2(c), respectively. It can be seen that the minimum-variance estimators outperform the HMM-based estimators when the model parameters are estimated from the available measurements.

The procedure described in above was used to identify the unknown  $F_N$  and  $Q_N$  from the initial state estimates (26). It was noticed that enforcing observability and reachability (32) reduced the number of quantisation levels to about  $n = 16$  at low SNRs. The developed filter (33) – (34) and smoother (35) – (36) were applied to the measurements. The RMS positioning errors exhibited by the developed filter and smoother are indicated by lines (iii) of Fig. 2(b) and Fig. 2(c), respectively. It can be seen that the developed filter and smoother can provide better than 0.1 m RMS error improvement over 12 to 20 dB SNR.

Thus, this example demonstrates that mean-square-error performance benefits can be attained by the developed filter and smoother. The benefit arises because the state-space model (20) – (21) includes a transition probability matrix, a residual  $Gw_i$  (which is absent within the HMM approaches of [8] - [10]), and they are minimum-variance designs.

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“Knowledge is not a series of self-consistent theories that converges toward an ideal view; it is rather an ever increasing ocean of mutually incompatible (and perhaps even incommensurable) alternatives, each single theory, each fairy tale, each myth that is part of the collection forcing the others into greater articulation and all of them contributing, via this process of competition, to the development of our consciousness.” *Paul Feyerabend*

## 11.5 High-Order-Minimum-Variance HMM Filtering and Smoothing

### 11.5.1 Overview

In this section, the objective is to investigate filter formulations which may be able to provide further performance improvements in applications where some repetitions within measurement trajectories occur. A high-order signal model is proposed, in which the states comprise Kronecker products of Markov chains. Kronecker products are routinely employed for filtering multi-dimensional signals [47] and constructing large Markov models [48]. The Kronecker-product states may be plugged into the previously-described predictor, filter and smoother recursions. Although employing Kronecker tensor products results in a nonlinear combination of probability distributions, a signal model parameterisation is described below which enables optimal linear filters to be constructed.

The inclusion of Markov chain states in an optimal filtering framework has been considered in the previous section. Here a signal model is proposed that possesses an additional level of abstraction or composition. In particular, an  $n$ -step Markov chain is described in which the states are Kronecker tensor products of  $n$  previous probability distributions and an  $m^n \times m^n$  stochastic matrix. This approach endeavours to capture latent interdependencies between signals' probability distributions in the pursuit of improved filter performance.

Utilising Kronecker products of states from multiple previous time steps can provide improved prediction performance. Selecting the number of previous time steps remains an open problem. It is shown here that residual error variances can be used to compare candidate filter designs.

The  $n$ -step Markov chain and an output estimation problem are defined in Section 11.5.2. An optimal filter that estimates an  $n$ -step Markov chain from noisy measurements is developed in Section 11.5.3. A disadvantage of the proposed approach is a significant increase in the number of calculations. It is advocated that  $m$  and  $n$  can be selected by searching for the filter that exhibits minimum residual error. Section 11.5.4 discusses a coal shiploader application. It is demonstrated that an  $n$ -step Markov chain, minimum-variance filter can outperform conventional Kalman and HMM filters for estimating the surfaces of coal being loaded into a ship's hold.

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"In the beginning there was nothing, which exploded." *Sir Terrence David John Pratchett*

## 11.5.2 Problem Definition

### 11.5.2.1 Kronecker-Product Signal Model

Consider a vector  $x_k = [x_{k,1}, x_{k,2}, \dots, x_{k,m}]^T \in \mathbb{R}^m$ , in which  $m$  is a positive integer. The Kronecker tensor product of integer  $n$  vectors  $x_k, x_{k-1}, \dots, x_{k-n+1}$ , for integer  $k \geq n+2$ , is denoted by

$$x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1} \in \mathbb{R}^{m^n}. \quad (39)$$

Operator precedence is omitted within (39) because the Kronecker tensor product is associative. It is convenient to denote the set of Kronecker products (39) over an interval of length  $N$  by

$$\mathcal{X} = [x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1}, \dots, x_N \otimes x_{N-1} \cdots \otimes x_{N-n+1}] \in \mathbb{R}^{m^n \times N}.$$

Similarly, let  $u_k = [u_{k,1}, u_{k,2}, \dots, u_{k,m}]^T \in \mathbb{R}^m$  and

$$\mathcal{U} = [u_k \otimes u_{k-1} \cdots \otimes u_{k-n+1}, \dots, u_N \otimes u_{N-1} \cdots \otimes u_{N-n+1}] \in \mathbb{R}^{m^n \times N}.$$

The inner product of  $\mathcal{X}$  and  $\mathcal{U}$  is defined as

$$\langle \mathcal{U}, \mathcal{X} \rangle = \sum_{k=n+2}^N (u_k \otimes u_{k-1} \cdots \otimes u_{k-n+1})^T (x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1}).$$

The 2-norm of  $\mathcal{U}$  is defined as  $\|\mathcal{U}\|_2 = \langle \mathcal{U}, \mathcal{U} \rangle$ .

The Doob-Meyer Decomposition Theorem [11] states that a stochastic process may be decomposed into the sum of two parts, namely, a prediction and an input process. Therefore, a linear system  $\mathcal{G}$  is postulated which operates on an input  $\mathcal{W} = [w_1, w_2, \dots, w_N] \in \mathbb{R}^{m^n \times N}$ ,  $w_t \in \mathbb{R}^{m^n \times 1}$ , and produces an output  $\mathcal{Y} = [y_1, y_2, \dots, y_N] \in \mathbb{R}^{1 \times N}$ ,  $y_k \in \mathbb{R}$ , i.e.,  $\mathcal{Y} = \mathcal{G}\mathcal{W}$ . It is assumed that  $\mathcal{G}$  has states realised by

$$x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1} = A^{(n)} x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2} + w_{k-1}, \quad (40)$$

---

“Not only is the Universe stranger than we think, it is stranger than we can think.” *Werner Karl Heisenberg*



where  $A^{(n)} \in \mathbb{R}^{m^n \times m^n}$  is a transition probability matrix and  $w_k$  is a time-homogenous stochastic input process with  $E\{\mathcal{W}_k \mathcal{W}_k^T\} = Q^{(n)}$ . It is also assumed that  $x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-1}$  are not observed directly but are hidden. The system output,  $y_k \in \mathbb{R}$ , is modelled as

$$y_{k-1} = C^{(n)} x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2}, \quad (41)$$

where  $C^{(n)} \in \mathbb{R}^{1 \times m^n}$ . For example, consider a second system  $\mathcal{H}$  of compatible dimensions that is driven by an input  $\mathcal{Z}$  and having an output  $\mathcal{U} = \mathcal{H}\mathcal{Z}$ . From the properties of Kronecker tensor products [45] it can be shown that  $\mathcal{J} \otimes \mathcal{U} = \mathcal{G} \mathcal{W} \otimes \mathcal{H}\mathcal{Z} = (\mathcal{G} \otimes \mathcal{H}) \mathcal{W} \otimes \mathcal{Z}$ . That is, the eigenvalues of  $\mathcal{G} \otimes \mathcal{H}$  arise as a product of the eigenvalues of  $\mathcal{G}$  and  $\mathcal{H}$  [45]. Thus, a system possessing Kronecker product states will be asymptotically stable if it is a product of asymptotically stable systems.

The adjoint of  $\mathcal{G}$ , denoted by  $\mathcal{G}^H$ , satisfies

$$\langle \mathcal{U}, \mathcal{G}\mathcal{W} \rangle = \langle \mathcal{G}^H \mathcal{U}, \mathcal{W} \rangle$$

for all  $\mathcal{U} \in \mathbb{R}^{1 \times T}$ , where  $(\cdot)^H$  denotes the adjoint operator. Adjoint systems are required in the optimal filter and smoother derivations of [12].

### 11.5.2.2 High-Order Markov Chains

**Definition 1:** Suppose there exists a probability distribution of a discrete-time vector  $x_k = [x_{k,1}, x_{k,2}, \dots, x_{k,m}]^T \in \mathbb{R}^m$  satisfying: (i)  $0 \leq x_{k,i} \leq 1$ ,  $i = 1, \dots, m$ ; and (ii)  $\sum_{i=1}^m x_{k,i} = 1$ .

Two useful observations about Kronecker products of probability distribution vectors are stated below.

**Lemma 6 [46]:** The elements of the vector  $x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1} \in \mathbb{R}^{m^n}$  satisfy the properties of a probability distribution vector.

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“Gravity explains the motions of the planets, but it cannot explain who sets the planets in motion.” *Sir Isaac Newton*

**Proof:** (i) The non-negativity property follows by inspection. (ii) It is easily verified that

$$\sum_{i=1}^m x_{k,i} \otimes x_{k-1,i} \cdots \otimes x_{k-n-1,i} = \sum_{i=1}^m x_{k,i} \sum_{i=1}^m x_{k-1,i} \cdots \sum_{i=1}^m x_{k-n-1,i} = 1.$$

**Lemma 7:** Let

$$C_1^{(n)} = \begin{bmatrix} 1,1,\dots,1 & 0,0,\dots,0 & & \\ 0,0,\dots,0 & 1,1,\dots,1 & & \\ & & \ddots & 0,0,\dots,0 \\ & & 0,0,\dots,0 & 1,1,\dots,1 \end{bmatrix} \in \mathbb{R}^{m \times m},$$

Then

$$x_k = C_1^{(n)} x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1}. \quad (42)$$

**Proof:** The claim can be verified by evaluating (42) and using  $\sum_{i=1}^m x_{k,i} = 1$ .

Without loss of generality, suppose that  $x_k, x_{k-1}, \dots, x_{k-n-1}$  within (40) are orthogonal basis vectors in  $\mathbb{R}^{m^n}$ . Let us now consider  $A^{(n)}$ . Post-multiplying (40) by  $(x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})^T$ , taking the expectation over  $k \in [1, N]$  and rearranging yields

$$A^{(n)} = E\{(x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1})(x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})^T\} \\ \times \left( E\{(x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})(x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})^T\} \right)^{-1}. \quad (43)$$

It is shown below that the  $(i,j)^{\text{th}}$  element of  $A^{(n)}$ , denoted by  $a_{i,j}^{(n)}$ , is the probability of transitioning from state  $x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2} = j$  to state  $x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1} = i$ .

**Lemma 8 [46]:** The above  $A^{(n)}$  is a column stochastic transition probability matrix.

**Proof:** Expanding (43) yields

---

"The mind likes a strange idea as little as the body likes a strange protein and resists it with similar energy. It would not perhaps be too fanciful to say that a new idea is the most quickly acting antigen known to science." *Wilfred Trotter*

$$A^{(n)} = NE\{(x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1})(x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})^T\} \\ \times \left( NE\{(x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})(x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})^T\} \right)^{-1},$$

in which

$$a_{i,j}^{(n)} = \frac{\sum_{k=n+1}^N (x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1})_i (x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})_j}{\sum_{t=n+1}^N (x_{t-1} \otimes x_{t-2} \cdots \otimes x_{t-n-2})_j^2} \\ = \frac{\sum_{k=n+1}^N (x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1})_i (x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})_j}{\sum_{k=n+1}^N (x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2})_j} \\ = Pr(x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1} = i \mid x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2} = j). \quad (44)$$

As a consequence of (44), the vector  $x_k \otimes x_{k-1} \cdots \otimes x_{k-n+1}$  within (40) is termed an  $n$ -step Markov chain. For example,  $x_k = A^{(1)}x_{k-1}$  and  $x_k \otimes x_{k-1} = A^{(2)}x_{k-1} \otimes x_{k-2}$  are called 1-step and 2-step Markov chains, respectively.

The representation (40), (41) defines a linear system in which  $x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2}$  is the state vector. It is assumed herein that the  $x_k$  within (40) are probability distribution vectors. From Lemma 8, the Kronecker products within (40), (41) are also probability distribution vectors.

### 11.5.2.3 Output Estimation

Suppose that signals are generated by the system (40), (41) and that observations

$$z_k = y_k + v_k, \quad (45)$$

are available, where  $v_t \in \mathbb{R}$  is a stationary measurement noise sequence with variance  $\sigma_v^2$ . It is desired to develop a causal solution  $\mathcal{H}$  which operates on measurements  $\mathcal{Z} = [z_1, z_2, \dots, z_k]$  and produces filtered estimates  $\hat{y}_{k/k}$  of  $y_k$ . The filter performance objective is to minimise the 2-norm of the causal part of the output estimation error, *i.e.*,

$$\| \{ \mathcal{Y} - \mathcal{H}\mathcal{Z} \}_+ \|_2, \quad (46)$$

---

“Millions saw the apple fall, Newton was the only one who asked why?” *Bernard Baruch*

where  $\{\cdot\}_+$  denotes the causal part.

### 11.5.3 Optimal Estimation of an $n$ -Step Markov Chain

#### 11.5.3.1 Filter Realisation

The optimal minimum-variance filter for the output estimation problem (40), (41), (45), (46) is given by

$$\mathcal{H} = \mathcal{G} \{ \mathcal{Q}^{(n)} \mathcal{G}^H (\mathcal{G} \mathcal{Q}^{(n)} \mathcal{G}^H + \sigma_v^2)^{-1} \}_+, \quad (47)$$

as shown in [12]. Let  $\hat{x}_{k/k-1} \otimes \hat{x}_{k-1/k-1} \cdots \otimes \hat{x}_{k-n-1/k-1}$  and  $\hat{x}_{k/k} \otimes \hat{x}_{k-1/k} \cdots \otimes \hat{x}_{k-n-1/k}$  denote the predicted and filtered estimates of  $x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1}$  at time  $k$ , respectively. These estimates are calculated as

$$\hat{x}_{k/k} \otimes \hat{x}_{k-1/k} \cdots \otimes \hat{x}_{k-n-1/k} = \hat{x}_{k/k-1} \otimes \hat{x}_{k-1/k-1} \cdots \otimes \hat{x}_{k-n-1/k-1} \quad (48)$$

$$+ L(z_k - C^{(n)} \hat{x}_{k/k-1} \otimes \hat{x}_{k-1/k-1} \cdots \otimes \hat{x}_{k-n-1/k-1} - E\{z_t\}), \quad (49)$$

$$\hat{x}_{k+1/k} \otimes \hat{x}_{k/k} \cdots \otimes \hat{x}_{k-n/k} = A^{(n)} \hat{x}_{k/k} \otimes \hat{x}_{k-1/k} \cdots \otimes \hat{x}_{k-n-1/k}$$

where  $L = P(C^{(n)})^T \Omega^{-1}$  is the filter gain,  $\Omega = C^{(n)} P (C^{(n)})^T + \sigma_v^2$ , in which  $P$  is the solution of the algebraic Riccati equation  $P = A^{(n)} P (A^{(n)})^T - A^{(n)} L C^{(n)} P (A^{(n)})^T + Q^{(n)}$ . The filtered output estimates are given by

$$\hat{y}_{k/k} = C^{(n)} \hat{x}_{k/k} \otimes \hat{x}_{k-1/k} \cdots \otimes \hat{x}_{k-n-1/k} + E\{z_t\} \quad (50)$$

The nonzero mean  $E\{z_k\}$  can (for example) be obtained by calculating a moving average alongside the filter recursions. It is asserted that the above output estimate satisfies the previously-stated filtering performance objective.

**Lemma 9:** The estimate (50) minimises  $\|\{\mathcal{Y} - \mathcal{H}\mathcal{Z}\}_+\|_2$ .

The proof is omitted since it follows *mutatis mutandis* from [12]. The above signal model could be similarly applied within smoothers (such as [12]) to obtain further performance benefits.

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“All that was required to measure the planet was a man with a stick and a brain. In other words, couple an intellect with some experimental apparatus and almost anything seems achievable.” *Simon Lehna Singh*

### 11.5.3.2 Design Procedure

It is assumed that initial estimates of  $x_k$  may be calculated as  $x_k = [x_{1,k}, \dots, x_{m,k}]^T = q(z_k)$ , where  $q: \mathbb{R} \rightarrow \mathbb{R}^m$  denotes a function which discretises  $y_k$  into  $m$  equally-spaced levels and produces estimates of  $x_k \in \mathbb{R}^m$ , *i.e.*,

$$x_{i,k} = \begin{cases} 1 & \text{if } c_i - \Delta/2 \leq z_k < c_i + \Delta/2 \\ 0 & \text{otherwise} \end{cases} \quad (51)$$

where

$$C_2 = [c_1, c_2, \dots, c_m] \in \mathbb{R}^{1 \times m} \quad (52)$$

is a row vector of discretisation level centers that are spaced  $\Delta$  apart. Estimates of the Kronecker tensor product states may then be calculated as  $x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1}$ .

A least-squares estimate of the transition probability matrix for  $n$  discretisation levels is given by [33]:

$$A^{(n)} = E\{(x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1})^T (x_{k-2} \otimes x_{k-2} \cdots \otimes x_{k-n-2})\} \\ \times E\{(x_{k-2} \otimes x_{k-2} \cdots \otimes x_{k-n-2})^T (x_{k-2} \otimes x_{k-2} \cdots \otimes x_{k-n-2})\}^{-1}. \quad (53)$$

It is tacitly assumed that the  $\mathcal{G}$  within the optimum solution (47) is causal or stable. However, the transition probability matrix (53) has one maximum eigenvalue of 1, which results in  $\mathcal{G}$  being marginally unstable. This suggests that the filter is at risk of being marginally unstable. A method for removing the potential instability is described in [23] and Lemma 1.

We seek to extract  $y_k$  from the Kronecker tensor product states  $x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1}$ . Since the elements of  $x_k$  pertain to mutually exclusive transitions, the outputs are calculated as the expected payoff, *i.e.*,  $y_k = C_2 x_k$ , where  $C_2$  is defined in (52). Thus, the output mapping can be calculated as

$$C^{(n)} = C_2 C_1^{(n)}, \quad (54)$$

---

“The fact that we live at the bottom of a deep gravity well, on the surface of a gas covered planet going around a nuclear fireball 90 million miles away and think this to be normal is obviously some indication of how skewed our perspective tends to be.” *Douglas Noel Adams*

where  $C_1^{(n)}$  is defined within Lemma 7. The above mapping affords a linear estimation problem which dispenses with the need of an extended Kalman filter for recovering  $y_t$  from the nonlinear combination of probability distributions within (2).

Since the assumed system (41), (42) is time-invariant, an input sequence covariance can be obtained from the steady-state estimate:  $Q^{(n)} = (I - A^{(n)})E\{(C^\dagger x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1}) \times (C^\dagger x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1})^T\}(I - A^{(n)})^T$ , where  $(.)^\dagger$  denotes the Moore-Penrose pseudo-inverse.

Guidance is desired for selecting the number of discretisations  $m$  and the number tensor products  $n$ . To this end, a method is suggested below for comparing candidate filter designs. Let  $\hat{A}^{(n)}$  and  $\hat{C}^{(n)}$  denote available estimates of the actual  $A^{(n)}$  and  $C^{(n)}$ , respectively.

**Lemma 10 [46]:** Let  $\tilde{x}_{k/k-1} \otimes \tilde{x}_{k-1/k-1} \cdots \otimes \tilde{x}_{k-n-1/k-1} = x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1} - \hat{x}_{k/k-1} \otimes \hat{x}_{k-1/k-1} \cdots \otimes \hat{x}_{k-n-1/k-1}$  and  $P = E\{(\tilde{x}_{k/k-1} \otimes \tilde{x}_{k-1/k-1} \cdots \otimes \tilde{x}_{k-n-1/k-1})(\tilde{x}_{k/k-1} \otimes \tilde{x}_{k-1/k-1} \cdots \otimes \tilde{x}_{k-n-1/k-1})^T\}$  respectively denote the state prediction error and its stationary covariance. If  $\hat{A}^{(n)} = A^{(n)}$  and  $\hat{C}^{(n)} = C^{(n)}$  the residual error variance

$$E\{y_{k/k-1} y_{k/k-1}^T\} = C^{(n)} P (C^{(n)})^T + R \quad (55)$$

is minimised.

**Proof:** Subtracting (49) from (40) and rearranging yields

$$\begin{aligned} & \tilde{x}_{k/k-1} \otimes \tilde{x}_{k-1/k-1} \cdots \otimes \tilde{x}_{k-n-1/k-1} \\ &= (A^{(n)} - KC^{(n)})\tilde{x}_{k-1/k-1} \otimes \tilde{x}_{k-2/k-1} \cdots \otimes \tilde{x}_{k-n-2/k-1} + Kv_t + w_t \\ &+ (A^{(n)} - \hat{A}^{(n)} - K(C^{(n)} - \hat{C}^{(n)}))x_{k-1/k-1} \otimes x_{k-2/k-1} \cdots \otimes x_{k-n-2/k-1}. \end{aligned}$$

Constructing the state prediction error covariance gives

$$\begin{aligned} P &= (A^{(n)} - KC^{(n)})P(A^{(n)} - KC^{(n)})^T + K\sigma_v^2 K^T + Q \\ &+ (A^{(n)} - \hat{A}^{(n)} - K(C^{(n)} - \hat{C}^{(n)}))P_2 \\ &\times (A^{(n)} - \hat{A}^{(n)} - K(C^{(n)} - \hat{C}^{(n)}))^T \\ &+ (A^{(n)} - KC^{(n)})P_3(A^{(n)} - \hat{A}^{(n)} - K(C^{(n)} - \hat{C}^{(n)}))^T \end{aligned}$$

---

“Its easy to come up with new ideas; the hard part is letting go of what worked for you two years ago, but will soon be out of date.” *Roger Von Oech*

$$+(A^{(n)} - \hat{A}^{(n)} - K(C^{(n)} - \hat{C}^{(n)}))P_3^T(A^{(n)} - KC^{(n)})^T, \quad (56)$$

where  $P_2 = E\{(x_{k/k-1} \otimes x_{k-1/k-1} \cdots \otimes x_{k-n-1/k-1})(x_{k/k-1} \otimes x_{k-1/k-1} \cdots \otimes x_{k-n-1/k-1})^T\}$  and  $P_3 = E\{(x_{k/k-1} \otimes x_{k-1/k-1} \cdots \otimes x_{k-n-1/k-1})(\tilde{x}_{k/k-1} \otimes \tilde{x}_{k-1/k-1} \cdots \otimes \tilde{x}_{k-n-1/k-1})^T\}$ . The claim follows by inspection of (56).

The filter error covariance is affine to the prediction error covariance (see [1]). Thus, the conditions of Lemma 10 also minimise the filter error. Suppose that signal models with different  $m$  and  $n$  have compatibly-dimensioned state space matrices which are padded with zeroes. Then Lemma 10 suggests that candidate filter designs could be assessed by comparing their residual error variances. The advantage of comparing residual error covariances is demonstrated within the example that follows.

### 11.5.4 Application

Assisting shiploading requires identification of the cargo hold of a ship and continuous estimation of the volume of coal residing therein. The application of a LIDAR for estimating the volume of loaded material is described in [50] – [54].

A SICK LD-LRS3611 LIDAR, was mounted on an EXLAR Tritex II rotator that was installed above the chute of an operational coal shiploader. The LIDAR was configured to scan a  $360^\circ$  field of view at  $0.25^\circ$  increments (*i.e.*, 1440 scans) and a 10 Hz rotation rate within the vertical plane. The EXLAR Tritex II rotator angles were varied from  $-35^\circ$  to  $+260^\circ$  at  $0.83^\circ$  increments (*i.e.*, 356 angles) in the horizontal plane. The  $356 \times 1440$  measurement records was stored in a MySQL database over 40-hour-long shiploading events and monitored remotely. An accumulated LIDAR point cloud model of a ship's hold during loading is shown in Fig. 3. A heap of coal can easily be identified in the centre of the figure.

Coal dust spills out of the chute and accumulates on the LIDAR lens which degrades the measurements. To demonstrate the efficacy of the described filter, independent Gaussian noise realisations were added to the LIDAR measurements. For an  $n$ -step,  $m$ -discretisation-level Markov model, an  $m^n \times m^n$  transition probability matrix needs to be identified from the data. The performance of the developed filter depends on the Markov model assumptions and the prevailing signal-to-noise ratio (SNR). The candidate assumptions considered here include 1-step, 2-step and 3-Step Markov chains together with  $m = 16$  discretisations. The above design procedures were used to implement filters that operate on the individual 356 vertical-plane measurements.

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“If the bee disappeared off the face of the earth, man would only have four years left to live.” *Maurice Maeterlinck, The Life of the Bee*

The 2-step Markov chain filter was found to provide the best residual error variance performance (see Lemma 5) over -10 to 10 dB SNR. It was observed that the transition probability matrix is approximately tri-diagonal, which suggests that about  $3\sqrt{2} \times 16^2$  transition probability matrix elements were estimated. It is known that the Cramer-Rao lower bound for estimating a parameter of a signal in white Gaussian noise is proportional to the noise variance [49]. Thus, signal complexity, SNR and calculation cost need to be considered in the design of  $\{m, n\}$ .

For comparison, a first-order Kalman filter was designed by assuming a model of the form

$$x_{k+1} = Fx_k + w_k, \quad (57)$$

$$z_k = x_k + v_k, \quad (58)$$

in which  $F, \sigma_w^2 \in \mathbb{R}$  were estimated from the measurements. The RMSE observed for the first-order Kalman filter versus SNR is indicated by the line (i) of Fig. 4. It can be seen that the minimum-error-residual 2-step Markov chain filter (line (ii)) performs better than the first order filter.

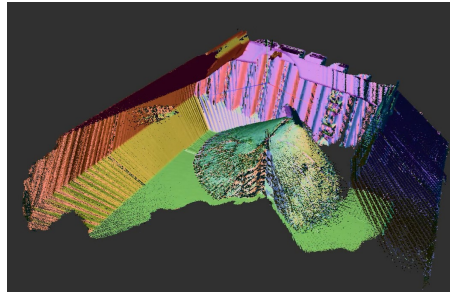


Fig. 3. Sample snapshot of a LIDAR point cloud of a ship's hold showing heaped coal.

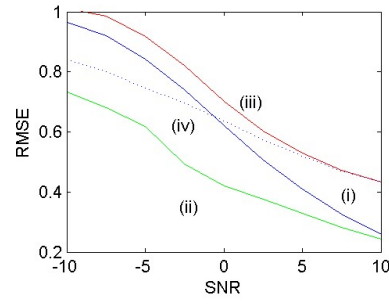


Fig. 4. Filter performance comparison: (i) first-order Kalman filter; (ii) minimum-error-residual 2-step Markov chain filter; (iii) HMM filter with  $a$  estimated from  $z_t$ ; (iv) HMM filter with  $a$  estimated from  $y_t$ .

In an HMM approach, the signal amplitude range are discretised into  $m$  levels centered around points  $q_i, i = 1 \dots m$ . An HMM filter and a fixed-interval HMM smoother were implemented using the techniques described in Section 11.3, namely, using (6) - (9), assuming Gaussian probability distributions (15) - (16). The performance exhibited a “blind” HMM filter, in which  $m = 16$  and  $F$  was estimated from the measurements  $z_t$ , is indicated by line (iii) of Fig. 4. Better



HMM filter performance was obtained by estimating  $a$  from the noiseless LIDAR signal  $y_k$ , as is indicated by line (iv) of Fig. 4.

It can be seen that the minimum-error-residual 2-step Markov chain filter outperforms the Kalman filter and the two HMM filters. Note that a conventional Kalman filter minimises the mean square error (MSE) but it does not exploit transition probabilities. Although a conventional HMM filter possesses a transition probability matrix, it does not explicitly minimise the MSE. The developed  $n$ -step Markov chain filter provides a performance benefit because it exploits transition probability information and is a minimum-MSE design. It also employs Kronecker tensor product states which allow interdependencies between signals' probability distributions to be captured. However, there is a significant increase in calculation cost, namely, the filters require  $(2m^n + 1)N$  inner products of  $m^n$ -dimensional quantities.

Note that an  $n$ -step Markov chain can similarly be used within fixed-lag and fixed-interval minimum-variance, HMM and minimum-variance-HMM smoothers. This is expected to provide incremental performance gains at the expense of further calculations.

## 11.6 Chapter Summary

This chapter shows how observed sequences of patterns may be exploited to improve signal recovery performance. To this end, HMM, minimum-variance HMM, and high-order-minimum-variance-HMM filter/smoothing derivations are set out. These estimators are summarised in Table 1.

It is illustrated with the aid of examples that HMM estimators can outperform the optimal minimum-variance solutions at low-SNR, provided that noiseless “training data” is available for parameter estimation. The examples also demonstrate that discretisation (or quantisation) degrades signal fidelity and performance at high SNR.

The minimum-variance-HMM estimators arise by employing transition probability matrices and probability distribution states within the standard minimum-variance filter/smoothing recursions. MSE benefits can arise because the estimators assume the presence of a process noise and are minimum-error-variance designs. However, the model order (or the number of discretisation/quantisation levels) needs to be optimised by the designer. A low-order model can compromise signal fidelity. If the model order is too high, the transition probability matrix can become rank deficient. Consequently, deflating the model to enforce observability and reachability is advocated.

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“I like thinking big. If you're going to be thinking anything, you might as well think big.” *Donald John Trump*

	ASSUMPTIONS	MAIN RESULTS
HMM	<p>Signal model</p> $x_k \in \{q_1, \dots, q_N\}, q_i \in \mathbb{R}$ $y_k = x_k + v_k$ <p>Transition probability matrix</p> $A = \{a_{ij}\} \in \mathbb{R}^{N \times N}$ $a_{i,j} = \Pr\{x_{k+1} = q_i \mid x_k = q_j\}$	<p>Filtered probability distributions</p> $\alpha_k(j) = C_k(i) \sum_{j=1}^N a_{j,i}(k) \alpha_{k-1}(j)$ <p>where <math>C_{k+1}(i) = \Pr\{y_{k+1} \mid x_{k+1} = i\}</math></p> <p>Filtered estimate</p> $i = \arg \max_{1 \leq i \leq N} \alpha_k(i), \hat{x}_k = q_i$ <p>Backward probability distributions</p> $\beta_k(j) = \sum_{j=1}^N \beta_{k+1}(j) C_{k+1}(j) a_{j,i}(k+1)$ <p>Smoothed probability distributions</p> $\gamma_k(i) = \frac{\beta_k(i) \alpha_k(i)}{\sum_{i=1}^N \beta_k(i) \alpha_k(i)}$ <p>Smoothed estimate</p> $i = \arg \max_{1 \leq i \leq N} \gamma_k(i), \hat{x}_{k/T} = q_i.$
Minimum-variance HMM	<p>Signal model</p> $x_{k+1} = F_n x_k + G u_k$ $y_k = H_n x_k + J w_k$ $\pi = F_n \pi$ $H_n = [h_1, \dots, h_n]$	<p>Filter recursions</p> $\hat{x}_{k/k} = \hat{x}_{k/k-1} + \bar{P} H_n^T \Omega^{-1} (Y_k - H_n \hat{x}_{k/k-1} - H_n \pi),$ $\hat{x}_{k+1/k} = F_n \hat{x}_{k/k}$ <p>Smoother recursions</p> $\alpha_k = \Omega^{-1/2} (Y_k - H_n \hat{x}_{k/k-1} - H_n \pi)$ $\xi_{k-1} = (F_n^T - H_n^T K^T) \xi_k + H_n^T \Omega^{-1/2} \alpha_k,$ $\xi_N = 0$ $\beta_k = -K^T \xi_k + \Omega^{-1/2} \alpha_k$ $\hat{y}_{k/N} = Y_k - J J^T \beta_k$
High-order-minimum-variance HMM	<p>Signal model</p> $x_k \otimes x_{k-1} \cdots \otimes x_{k-n-1}$ $= A^{(n)} x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2} + w_{k-1}$ $y_{k-1} = C^{(n)} x_{k-1} \otimes x_{k-2} \cdots \otimes x_{k-n-2}$ $z_k = y_k + v_k$ $C_1^{(n)} = \begin{bmatrix} 1, 1, \dots, 1 & 0, 0, \dots, 0 \\ 0, 0, \dots, 0 & 1, 1, \dots, 1 & & \\ & & \ddots & 0, 0, \dots, 0 \\ & & & 0, 0, \dots, 0 & 1, 1, \dots, 1 \end{bmatrix}$ $C_2 = [c_1, c_2, \dots, c_m]$ $C^{(n)} = C_2 C_1^{(n)}$	<p>Similar to above application of the minimum-variance filter and smoother recursions to the signal model</p>

Table 1. Summary of HMM, minimum-variance HMM and high-order-minimum-variance HMM filter/smoothing recursions.

“The whole [scientific] process resembles biological evolution. A problem is like an ecological niche, and a theory is like a gene or a species which is being tested for viability in that niche.” *David Deutsch*

Optimal prediction conventionally involves operating on the state vector from one previous time step. Although it may be appealing to contemplate higher-dimensional matrix algebra to generalise predictions from multiple previous time steps, formulating the associated Riccati equations would become unwieldy. A more elegant and tractable approach is to simply work with Kronecker tensor products of previous states. Indeed, this is done in the development of the so-called high-order-HMM filter/smoother. This approach allows more transition probability interdependencies to be captured and exploited.

If the applications at hand are insensitive to calculation cost and suboptimal estimation performance is acceptable then experimenting with arbitrary parameter values within the minimum-variance solutions may suffice. It may be possible to reduce the MSE by instead employing the well-known least-squares (or better, unbiased-least-squares) parameter estimates. Conversely, if performance is paramount and higher calculation overheads can be tolerated then including transition probability matrices and Kronecker products of probability distribution states within minimum-variance recursions (as described above) can be considered.

## 11.7 Problems

**Problem 1.** Calculate the transition probability matrices for Markov processes having the following sequences.

- (a)  $x_1=2, x_2=2, x_3=1, x_4=1, x_5=1, x_6=2, x_7=2, x_8=1, x_9=1, x_{10}=1$ .
- (b)  $x_1=1, x_2=2, x_3=3, x_4=1, x_5=2, x_6=3, x_7=1, x_8=2, x_9=3$ .
- (c)  $x_1=1, x_2=2, x_3=3, x_4=2, x_5=1, x_6=2, x_7=3, x_8=2, x_9=1$ .

**Problem 2.** Suppose that a Markov process  $x_k$  takes on index values in the set

$\{1,2,3\}$  and its transition probability matrix is  $A = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$ . For an initial

state  $x_1=3$ , calculate the probability that  $x_2=3, x_3=3, x_4=1, x_5=1, x_6=3, x_7=2$  and  $x_8=1$ .

**Problem 3.** Consider the transition matrix  $A = \begin{bmatrix} r & 1-r \\ 1-r & r \end{bmatrix}$ , where  $0 \leq r \leq 1$ . Prove

by induction that the combined transition matrix after  $m$  steps is

$$A^m = \begin{bmatrix} 0.5 + 0.5(2r-1)^m & 0.5 - 0.5(2r-1)^m \\ 0.5 - 0.5(2r-1)^m & 0.5 + 0.5(2r-1)^m \end{bmatrix}.$$

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“I think therefore I am.” *René Descartes*“

**Problem 4.** Consider the transition matrix  $A = \begin{bmatrix} 1-r & s \\ r & 1-s \end{bmatrix}$ , where  $0 \leq r, s \leq 1$ . Find the associated stationary probability distribution  $\pi$  that satisfies  $\pi = A\pi$ .

**Problem 5.** Derive expressions for **(a)**  $\alpha_k(i) = \Pr\{x_k = i, y_1, \dots, y_k\}$ ; **(b)**  $\beta_k(i) = \Pr\{y_{k+1}, \dots, y_T \mid x_k = i\}$  and **(c)**  $\gamma_k(i) = \sum_{j=1}^n \Pr\{x_k = i \mid y_1, \dots, y_T\}$ .

**Problem 6. (a)** Derive an expression for the one-step-ahead prediction error covariance in terms of an estimate  $\hat{A}$  of  $A$  and an estimate  $\hat{C}$  of  $C$ .  
**(b)** Show that the prediction error covariance is minimised at  $\hat{A} = A$  and  $C = \hat{C}$ .

## 11.8 Glossary

$\pi_k$	Probability distribution vector (state) at time $k$ , which satisfies $\pi_k = A\pi_{k-1}$ , where $A$ is a transition probability matrix.
$\pi$	Stationary probability distribution vector state at time $k$ , which satisfies $\pi = A\pi$ .
$\alpha_k(i)$	$i^{\text{th}}$ component of the (forward) probability distribution vector at time $k$ .
$C_k(i)$	$i^{\text{th}}$ component of the conditional observation likelihood at time $k$ .
$\beta_k(i)$	$i^{\text{th}}$ component of the backward probability distribution vector at time $k$ .
$\gamma_k(i)$	$i^{\text{th}}$ component of the smoothed probability distribution vector at time $k$ .
$O(F_n, H_n)$	Observability matrix, where $F_n$ and $H_n$ respectively denote $n^{\text{th}}$ -order probability transition matrix and output mapping.
$O(F_n^T, H_n^T)^T$	Reachability matrix.
$x_k \otimes x_{k-1}$	Kronecker tensor product of $x_k$ and $x_{k-1}$ .

## 11.9 References

- [1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1979.
- [2] C. K. Chui and G. Chen, *Kalman Filtering with Real-Time Applications*, Third Edition, Springer-Verlag, Berlin, 1999.
- [3] T. Söderström. *Discrete-time Stochastic Systems*, Springer –Verlag London Limited, 2002.
- [4] G. P. Basharin, A. N. Naumov, “The life and work of A. A. Markov”, *Linear Algebra and its Applications*, vol. 386, pp. 3 – 26, 2004.

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“Judge a man by his questions rather than by his answers.” *François-Marie Arouet aka Voltaire*

- [5] L. Baum and T. Petrie, “Statistical inference for probabilistic functions of finite state Markov chains”, *Ann. Math Stat.*, vol. 37, pp. 1554 – 1563, 1966.
- [6] J. K. Baker, “The dragon system – An overview”, *IEEE Acoustics, Speech and Signal Processing*, vol. 23, no. 1, pp. 24 – 29, 1975.
- [7] F. Jelinek, “A fast sequential decoding algorithm using a stack”, *IBM Research Developments*, vol. 13, pp. 675 – 685, 1969.
- [8] L. R. Rabiner, “A tutorial on hidden Markov models and selected applications in speech recognition”, *Proceeding of the IEEE*, vol. 77, no. 2, pp. 257 – 286, 1989.
- [9] V. Krishnamurthy and J. B. Moore, “On-Line Estimation of Hidden Markov Model Parameters Based on the Kullback-Leibler Information Measure”, *IEEE Trans. Signal Processing*, vol. 41, no. 8, Aug. 1993.
- [10] Y. Ephraim and N. Merhav, “Hidden Markov Processes”, *IEEE Trans. Information Theory*, vol. 48, no. 6, pp. 1518 – 1569, 2002.
- [11] P.-A. Meyer, “A decomposition theorem for supermartingales”, *Illinois J. Math.*, vol. 6, pp. 193 – 205, 1962.
- [12] G. A. Einicke, “Optimal and Robust Noncausal Filter Formulations”, *IEEE Trans. Signal Processing*, vol. 54, no. 3, pp. 1069 – 1077, Mar. 2006.
- [13] G. A. Einicke, G. Falco and J. Y. Malos, “Bounded Constrained Filtering for GPS/INS Integration”, *IEEE Trans. Automatic Control*, vol. 58, no. 1, pp. 125–133, 2013.
- [14] I. B. Collings and J. B. Moore, “An adaptive hidden Markov model approach to FM and M-ary DPSK demodulation in noisy fading channels”, *Signal Processing*, vol. 47, pp. 71 – 84, 1995.
- [15] R. J. Elliot, L. Aggoun and J. B. Moore, *Hidden Markov Models: Estimation and Control (Stochastic Modelling and Applied Probability)*, Springer-Verlag, New York, 1995.
- [16] L. Johnston and V. Krishnamurthy, “Hidden Markov model algorithms for narrowband interference suppression in CDMA spread spectrum systems”, *Signal Processing*, vol. 79, pp. 315 – 324, 1999.
- [17] M. W. Hofbaur and B. C. Williams, “Mode Estimation of Probabilistic Hybrid Systems” in C. Tomlin, M. Greenstreet (eds.), *Hybrid Systems: Computation and Control, HSC 2002, Lecture Notes in Computer Science*, Springer Verlag, Berlin, vol. 2289, pp. 253 – 266, 2002.
- [18] K. Y. Lee, S. McLaughlin and K. Shirai, “Speech enhancement based on neural predictive hidden Markov model”, *Signal Processing*, vol. 65, pp. 373 – 381, 1998.
- [19] K. Y. Lee and J. Rheem, “Smoothing approach using forward-backward Kalman filter with Markov switching parameters for speech enhancement”, *Signal Processing*, vol. 80, pp. 2579 – 2588, 2000.
- [20] Q. Yan, S. Vaseghi, E. Zavarehei, B. Milner, J. Darch, P. White and I. Andrianakis, “Format tracking linear prediction mode using HMMs and Kalman filters for noisy speech processing”, *Computer Speech and Language*, vol. 21, pp. 543 – 561, 2007.
- [21] M. Anisetti, C. A. Ardagna, V. Bellandi, E. Damiani and S. Reale, “Advanced Localization of Mobile Terminal in Cellular Network”, *Int. J. Communications, Networks and Systems Sciences*, vol. 1, pp. 95 - 103, 2008.

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“The alternative to thinking in evolutionary terms is not to think at all.” *Sir Peter Medawar*

- [22] S. Vasuhi and V. Vaidehi, "Target Detection and Tracking for Video Surveillance", *WSEAS Trans. Signal Processing*, vol. 10, pp. 168 – 177, 2014.
- [23] L. B. White, "A Comparison Between Optimal and Kalman Filtering for Hidden Markov Processes", *IEEE Signal Processing Letters*, vol. 5, no. 5, pp. 124 – 126, May 1998.
- [24] H. Hjalmarsson and B. Ninness, "Fast, non-iterative estimation of hidden Markov models", *Proc. IEEE Conf. Acoustics, Speech and Signal Processing (ICASSP'98)*, pp. 2253 - 2256, 1998.
- [25] S. Andersson, T. Ryden and R. Johansson, "Linear optimal prediction and innovations representations of hidden Markov models", *Stochastic Processes and their Applications*, vol. 108, pp. 131 - 149, 2003.
- [26] S. Andersson and T. Ryden, "Subspace Estimation and Prediction Methods for Hidden Markov Models", *Annals of Statistics*, vol. 37, no. 6B pp. 4131 - 4152, 2009.
- [27] B. Widrow, I. Kollar and M. – C. Liu, "Statistical Theory of Quantization", *IEEE Trans. Instrum. Meas.*, vol. 45, no. 2, pp. 353 - 361, Apr. 1996.
- [28] A. R. Liu and R. R. Bitmead, "Stochastic observability in network state estimation and control", *Automatica*, vol. 47, pp. 65 - 78, 2011.
- [29] B. Widrow and I. Kollar, *Quantization Noise: Roundoff Error in Digital Computation, Signal Processing, Control, and Communications*, Cambridge University Press, 2008.
- [30] L. Y. Wang and G. G. Yin, "Asymptotically efficient parameter estimation using quantized output observations", *Automatica*, vol. 43, pp. 1178 – 1191, 2007.
- [31] J. C. Agüero, G. C. Goodwin, and J.I. Yuz, "System identification using quantized data", *Proc. IEEE Conf. Decision and Control*, pp. 4263 – 4268, New Orleans, LA, Dec. 2007.
- [32] F. Gustafsson and R. Karlsson, "Statistical results for system identification based on quantized observations", *Automatica*, vol. 45, pp. 2794 – 2801, December 2009.
- [33] J. J. Ford and J. B. Moore, "Adaptive Estimation of HMM Transition Probabilities", *IEEE Trans. Signal Processing*, vol. 46, no. 5, pp. 1374 – 1385, May 1998.
- [34] Moon, T. K., "The Expectation-Maximization Algorithm", *IEEE Signal Processing Magazine*, vol. 13, pp. 47 – 60, Nov. 1996.
- [35] G. A. Einicke, G. Falco and J. T. Malos, "EM Algorithm State Matrix Estimation for Navigation", *IEEE Signal Processing Letters*, vol. 17, no. 5, pp. 437 – 440, May 2010.
- [36] P. Van Overschee and B. De Moor, "A Unifying Theorem for Three Subspace System Identification Algorithms", *Automatica*, vol. 31, no. 12, pp. 1853 - 1864, Dec. 1995.
- [37] D. Hsu, S. M. Kakade and T. Zhang, "A spectral algorithm for learning Hidden Markov Models", *J. Computer and System Sciences*, vol. 78, pp. 1460 – 1480, 2012.
- [38] L. Song, B. Boots, S. M. Siddiqi, G. J. Gordon and A. Smola, "Hilbert Space

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"The well-being of a neuron depends on its ability to communicate with other neurons. Studies have shown that electrical and chemical stimulation from both a neuron's inputs and its targets support vital cellular processes. Neurons unable to connect effectively with other neurons atrophy. Useless, an abandoned neuron will die." *Lisa Genova*

- Embeddings of Hidden Markov Models”, *Proc. 27<sup>th</sup> Int. Conf. on Machine Learning (ICML-10)*, Haifa, Israel, pp. 991 – 998, 2010.
- [39] S. M. Siddiqi, B. Boot and G. J. Gordon, “Reduced-Rank Hidden Markov Models”, *Proc. 13<sup>th</sup> Int. Conf. Artificial Intelligence and Statistics (AISTATS)*, Sardinia, Italy, pp. 741 – 748, 2010.
- [40] Y. Subasi and M. Demirekler, “Quantitative measure of observability for linear stochastic systems”, *Automatica*, vol. 50, pp. 1669 - 1674, 2014.
- [41] W. J. Stewart, *Probability, Markov Chains, Queues and Simulation: The Mathematical Basis of Performance Modeling*, Princeton University Press, New Jersey, 2009.
- [42] C. E. De Souza, “On Stabilizing Properties of Solutions of the Riccati Difference Equation”, *IEEE Trans. Automat. Control*, vol. 34, no. 12, pp. 1313 – 1316, Dec. 1989.
- [43] T. Kailath, *Linear Systems*, Prentice-Hall Inc. Englewood Cliffs, N.J., 1980.
- [44] P.-A. Meyer, "A Decomposition theorem for supermartingales". *Illinois Journal of Mathematics*, vol 6, pp. 193–205, 1962.
- [45] A. N. Langville and W. J. Stewart, “The Kronecker product and stochastic automata networks”, *J. of Computational and Applied Mathematics*, vol. 167, pp. 429 – 447, 2004.
- [46] G. A. Einicke, “High-Order of LIDAR Data to Assist Coal Shiploading”, *IEEE Transactions on Aerospace and Electronic Systems*, vol. 53, no. 3, Jun. 2017.
- [47] A. N. Delopoulos and S. D. Kollias, “Optimal filter banks for signal reconstruction from noisy components”, *IEEE Transactions on Signal Processing*, vol. 44, iss. 2, pp. 212 – 224, Feb. 1996.
- [48] P. Buchholz and P. Kemper, “Kronecker Based matrix Representations for large Markov Models”, in C. Baier et al. (eds): *Validation of Stochastic Systems*, Springer-verlag Berlin Heidelberg, pp. 256 – 295, 2004.
- [49] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1993.
- [50] E. Duff, "Automated Volume Measurement of Haul-Truck Loads", *Proc. Australian Conference on Robotics and Automation*, Melbourne Australia. 2000.
- [51] A. O. Hargrave and M. E. Bialkowski, "Scanning radar system for machine guidance," *Proc. 18<sup>th</sup> Conference on Mechatronics and Machine Vision in Practice*, Brisbane, 2011.
- [52] M. Chao, L. Haiwei, Z. Ning and S. Yang, “A Ship Cargo Hold Inspection Approach Using Laser Vision Systems”, *Telekomnika*, vol. 11, no. 1, pp. 339 – 337, Jan. 2013.<sup>40</sup>
- [53] M. Chao, L. Haiwei, Z. Ning and S. Yang, “3D Ship Shape Inspection Approach using Laser Vision Systems for Automatic Ship Loader”, *Int. Journal of Advancements in Computing Technology (IJACT)*, vol. 5, no. 7, Apr. 2013.
- [54] M. Chao, Y. Huang, H. Liu, Y. Shen and W. Mi, “Study on Target Detection & Recognition Using Laser 3D Vision Systems for Automatic Ship Loader”, *Sensors & Transducers*, vol. 158, iss. 11, Nov. 2013.

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“We can allow satellites, planets, suns, universe, nay whole systems of universe, to be governed by laws, but the smallest insect, we wish to be created at once by special act.” *Charles Robert Darwin*